Applied use of the Banach contraction principle: 1st order differential equation

Th (Rolle)

$f : [a,b] \to \mathbb{R}$ continuous,
differentiable on (a,b)

$f(a) = f(b)$

$\Rightarrow \exists c \in (a,b) : f'(c) = 0$

Proof

Assume, w.l.g., that f is not a constant function.

As f is continuous on $[a,b]$,

$$\exists c \in [a,b] : f(c) = \sup_{x \in [a,b]} f(x) ;$$

$$\exists d \in [a,b] : f(d) = \inf_{x \in [a,b]} f(x) .$$

Assume, w.l.g., that $f(a) = f(b) < f(c)$.

Clearly, $a < c < b$.

It holds that $f(x) \leq f(c) \quad \forall x \in [a,b]$. — (*)

As f is differentiable on (a,b),

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c).$$

From (*), $f'(c) = \lim_{x \downarrow c} \dfrac{f(x) - f(c)}{x - c} \leq 0$;

$$f'(c) = \lim_{x \uparrow c} \frac{f(x) - f(c)}{x - c} \geq 0 .$$

Hence, we obtain $f'(c) = 0$.

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Th

$$\boxed{\begin{array}{l} f:[a,b]\to\mathbb{R} \text{ continuous} \\ \qquad\qquad \text{differentiable on } (a,b) \\ \Longrightarrow \exists c\in(a,b):\ f'(c)=\dfrac{f(b)-f(a)}{b-a} \end{array}}$$

Proof

Let $k \equiv \dfrac{f(b)-f(a)}{b-a} \in \mathbb{R}$.

Using this k, define $F:[a,b]\to\mathbb{R}$ as follows:

$$F(x)=f(x)-f(a)-k(x-a) \quad \forall x\in[a,b].$$

Then, F is $\left\{\begin{array}{l}\text{continuous on } [a,b] \\ \text{differentiable on } (a,b).\end{array}\right.$

Furthermore, $F(a)=F(b)=0$.

From Th (Rolle),

$$\exists c\in(a,b):\ F'(c)=0$$

$$\therefore\ f'(c)=k=\frac{f(b)-f(a)}{b-a}.$$

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※ the mean value theorem

Application

$$\boxed{\begin{array}{l} f'(x) > 0 \quad \forall x \in I \qquad \text{— } (*) \\ \Rightarrow f : \text{strictly monotone increasing} \end{array}}$$

Proof

Let $x_1, x_2 \in I : x_1 < x_2$. — $(**)$

We show that $f(x_1) < f(x_2)$.

From the mean value theorem,

$$\exists c \in (x_1, x_2) : \quad f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Consequently,

$$f(x_2) - f(x_1) = \underbrace{f'(c)}_{>0 \ (*)} \underbrace{(x_2 - x_1)}_{>0 \ (**)} > 0.$$

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$$\boxed{\begin{aligned} &D \subset \mathbb{R}^2 \neq \emptyset, \text{ convex} \\ &f: D \to \mathbb{R} \\ &\frac{\partial f}{\partial y}(x,y) > 0 \quad \forall (x,y) \in D \\ &\Longrightarrow \forall (x,y_1), (x,y_2) \in D : y_1 < y_2, \\ &\qquad f(x,y_1) < f(x,y_2) \end{aligned}}$$

<u>Proof</u>

From <u>the mean value theorem</u>,

$$\exists c \in (y_1, y_2) : \frac{\partial f}{\partial y}(x,c) = \frac{f(x,y_1) - f(x,y_2)}{y_1 - y_2}.$$

Therefore,

$$f(x,y_1) - f(x,y_2)$$

$$= \underbrace{\frac{\partial f}{\partial y}(x,c)}_{>0} \cdot \underbrace{(y_1 - y_2)}_{<0} < 0.$$

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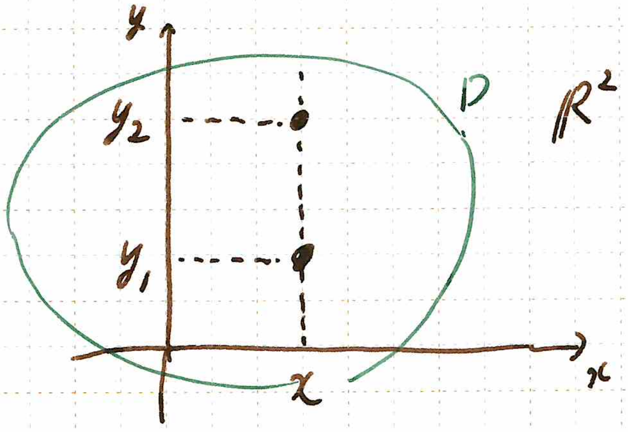

Th

$S \neq \phi$

$B(S) = \{ f : S \to \mathbb{R} \mid f \text{ is bdd.} \}$

$\|f\| = \sup_{t \in S} |f(t)| \quad \forall f \in B(S)$

$\Rightarrow (B(S), \|\cdot\|)$ Banach space

Proof (completeness)

Let $\{f_n\} \subset B(S)$ be a Cauchy seq. — ①

We prove that

$\exists f \in B(S) : \|f_n - f\| \to 0$

i.e. $\sup_{t \in S} |f_n(t) - f(t)| \to 0$

From ①, $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : \forall m, n \geq n_0, \|f_m - f_n\| < \varepsilon.$

i.e. $\forall t \in S, |f_m(t) - f_n(t)| < \varepsilon.$ — ②

$\therefore \forall t \in S, \varepsilon > 0, \exists n_0 \in \mathbb{N} : \forall m, n \geq n_0, |f_m(t) - f_n(t)| < \varepsilon.$

$\therefore \forall t \in S,$ $\{f_n(t)\}$ is a Cauchy seq. in $\mathbb{R}$.

As $\mathbb{R}$ is complete, $\forall t \in S, \exists f(t) \in \mathbb{R} : f_n(t) \to f(t).$ — ③

We have obtained $f : S \to \mathbb{R}$.

$\sup_{t \in S} |f_n(t) - f(t)| \to 0$ — ④

Let $\varepsilon > 0$.

From ②, $\exists n_0 \in \mathbb{N} : \forall m, n \geqq n_0, \forall t \in S,$

$$|f_m(t) - f_n(t)| < \varepsilon \quad (\forall m \geqq n_0).$$

As $m \to \infty$, we have from ③ that

$$|f(t) - f_n(t)| \leqq \varepsilon \quad (\forall t \in S)$$

Thus, $\sup_{t \in S} |f(t) - f_n(t)| \leqq \varepsilon$. — ⑤

$\therefore \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : \forall n \geqq n_0, \sup_{t \in S} |f_n(t) - f(t)| \leqq \varepsilon < 2\varepsilon$.

$f \in B(S)$

i.e. $f : S \to \mathbb{R}$ is bdd.

It follows that

$$\sup_{t \in S} |f(t)|$$

$$\leqq \sup_{t \in S} \{ |f(t) - f_{n_0}(t)| + |f_{n_0}(t)| \}$$

$$\leqq \sup_{t \in S} |f(t) - f_{n_0}(t)| + \sup_{t \in S} |f_{n_0}(t)|$$ ⑤

$$\leqq \varepsilon + \|f_{n_0}\|.$$

Hence, ④ means that $\|f_n - f\| \to 0$.

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Th

S MS

$f_n : S \to \mathbb{R}$ continuous $(n \in \mathbb{N})$

$f : S \to \mathbb{R}$

$\sup_{t \in S} |f_n(t) - f(t)| \to 0$ —(*)

$\Longrightarrow f$: continuous

Proof

We prove that

$\forall t_0 \in S, \varepsilon > 0, \exists \delta > 0 :$

$$d(t, t_0) < \delta \Rightarrow |f(t) - f(t_0)| < \varepsilon.$$

Let $t_0 \in S$ and $\varepsilon > 0$.

From (*), for $\varepsilon/3 > 0$,

$\exists n_0 \in \mathbb{N} : n \geq n_0 \Rightarrow \sup_{t \in S} |f_n(t) - f(t)| < \frac{\varepsilon}{3}.$ —(**)

As f_{n_0} is continuous (at t_0), for $\varepsilon/3 > 0$,

$\exists \delta > 0 : d(t, t_0) < \delta \Rightarrow |f_{n_0}(t) - f_{n_0}(t_0)| < \frac{\varepsilon}{3}.$ —(*3).

Let $t \in S : d(t, t_0) < \delta$.

Then, $|f(t) - f(t_0)|$

$$\leq |f(t) - f_{n_0}(t)| + |f_{n_0}(t) - f_{n_0}(t_0)| + |f_{n_0}(t_0) - f(t_0)|$$

(**) (*3)

$< \varepsilon.$

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Th

S MS

$B(S)$: Banach space with the sup-norm

$C(S) = \{f \in B(S) \mid f \text{ is continuous.}\}$

$\Rightarrow C(S)$ is closed in $B(S)$.

Proof.

Let $\{f_n\} \subset C(S) : f_n \to f \in B(S)$.

We show that $\underline{f \in C(S)}$.

OK.

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Th

S compact MS

$C(S) = \{f : S \to \mathbb{R} \mid f \text{ is continuous.}\} \ (\subset B(S))$

$\Rightarrow C(S)$ is closed in $B(S)$

Cor

S compact MS

$C(S) = \{f : S \to \mathbb{R} \mid f \text{ is continuous}\}$

$\Rightarrow C(S)$: complete

Cor

S compact MS

$X \subset C(S)$

$\Rightarrow$ Equivalent

① X : complete

② X : closed in $C(S)$.

Th A

$f:[a,b]\to\mathbb{R}$ continuous

$F(x)=\int_a^x f(t)\,dt \quad \forall x\in[a,b]$

$\Rightarrow F'=f$

Th B

$f:[a,b]\to\mathbb{R}$ continuous

$F'=f$

$\Rightarrow \int_a^b f(x)\,dx = F(b)-F(a)$

$D \subset \mathbb{R}^2$

$f : D \to \mathbb{R}$ continuous

$(x_0, y_0) \in D$

$I \subset \mathbb{R} : x_0 \in I$

$y : I \to \mathbb{R}$

$\forall x \in I, \ (x, f(x, y(x))) \in D$

$\Longrightarrow$ Equivalent

① $\begin{cases} y'(x) = f(x, y(x)) & \forall x \in I \\ y(x_0) = y_0 \end{cases}$

② $y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt \quad \forall x \in I$

$\underline{\underline{Th}}$

$x_0, y_0 \in \mathbb{R}$, $a, b > 0$

$D = \{(x,y) \in \mathbb{R}^2 \mid |x - x_0| \leqq a, |y - y_0| \leqq b\}$

$f, \dfrac{\partial f}{\partial y} : D \to \mathbb{R}$ continuous

$\Rightarrow \exists^1 y : I \to \mathbb{R} : \begin{cases} y'(x) = f(x, y(x)) & \forall x \in I \\ y(x_0) = y_0 \end{cases}$

where $I = [x_0 - h, x_0 + h]$ for some $h > 0$

Proof

We show that

$\exists^1 y : I \to \mathbb{R} : y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt \quad (\forall x \in I)$

where $I = [x_0 - h, x_0 + h]$

As f and $\frac{\partial}{\partial y} f(x,y)$ are continuous,

$\exists K > 0, M > 0$

$|f(x,y)| \leqq K$, — ①

$\left| \dfrac{\partial f}{\partial y}(x,y) \right| \leqq M \quad \forall (x,y) \in D$. — ②

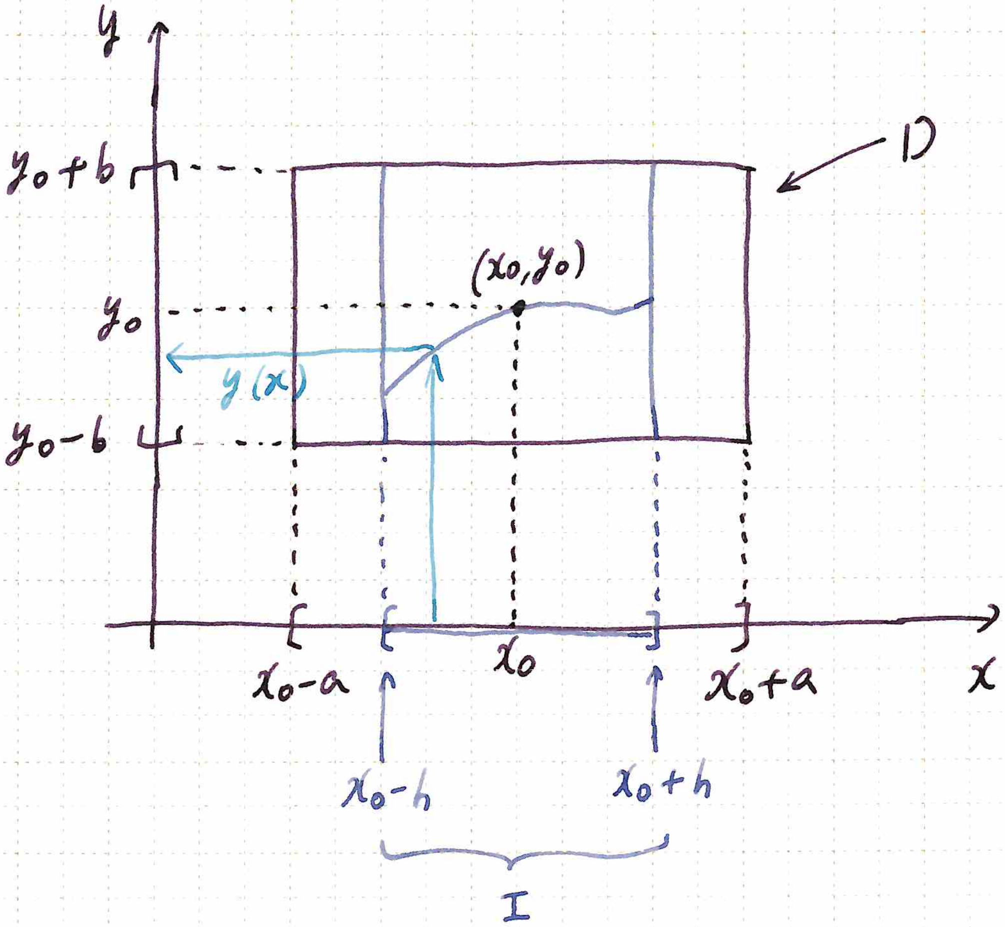
Let $(x, y_1), (x, y_2) \in D : y_1 < y_2$.

From the mean value theorem,

$\exists c \in (y_1, y_2)$:

$$|f(x, y_1) - f(x, y_2)| = \left| \frac{\partial f}{\partial y}(x, c) \right| |y_1 - y_2|$$

$$\leqq M |y_1 - y_2| \quad — ③$$


y
D
y0+b
(x0,y0)
y0
y(x)
y0-b
x0-a
x0
x0+a
x
x0-h
x0+h
I

Let $h \in \mathbb{R} : 0 < h < \min\{a, \frac{b}{K}, \frac{1}{M}\}$, and define $I \equiv [x_0 - h, x_0 + h]$.

Define $D' \equiv \{(x,y) \in \mathbb{R}^2 \mid |x - x_0| \leq h, |y - y_0| \leq Kh\} \subset D$.

Define $X = \{p \in C(I) \mid \forall x \in I, (x, p(x)) \in D'\}$

$\qquad = \{p \in C(I) \mid \forall x \in I, |p(x) - y_0| \leq Kh\}$.

Then, X is closed in $C(I)$. Furthermore, $X \neq \emptyset$.

As X is complete MS with the sup-norm, X is also complete.

Next, <u>we define a contraction mapping $T : X \to X$.</u>

Let $g \in X$.

i.e. $\forall t \in I, (t, g(t)) \in D' \subset D$ —④

Define $Tg \in C(I)$ as follows:

$$(Tg)(x) = y_0 + \int_{x_0}^{x} f(t, g(t))\,dt \quad \forall x \in I. \quad \text{—⑤}$$

Note that from ④, ⑤ is properly defined.

We show that <u>$Tg \in X$.</u>

For $g \in X$ and $x \in I$,

$$\left.\begin{aligned} |Tg(x) - y_0| &= \left| \int_{x_0}^{x} f(t, g(t))\,dt \right| \\ &\leq K|x - x_0| \\ &\leq Kh. \end{aligned}\right\} ①$$

$T: X \to X$ Mh-contraction $(0 < Mh < 1)$

i.e. $\forall g_1, g_2 \in X \ (\subset C(I))$,

$$\|Tg_1 - Tg_2\| \leq Mh \|g_1 - g_2\|$$

i.e. $\forall g_1, g_2 \in X$, $\forall x \in I$,

$$|Tg_1(x) - Tg_2(x)| \leq Mh \|g_1 - g_2\|$$

Let $g_1, g_2 \in X$ and $x \in I$.

i.e. $\forall t \in I, (t, g_1(t)), (t, g_2(t)) \in D' \subset D$ — ⑥

It follows that

$$|Tg_1(x) - Tg_2(x)|$$

$$= \left| \int_{x_0}^{x} f(t, g_1(t))\,dt - \int_{x_0}^{x} f(t, g_2(t))\,dt \right| \quad ⑤$$

$$= \left| \int_{x_0}^{x} \{ f(t, g_1(t)) - f(t, g_2(t)) \}\,dt \right| \quad ③ ⑥$$

$$\leq \left| \int_{x_0}^{x} M |g_1(t) - g_2(t)|\,dt \right|$$

$$\leq Mh \cdot \sup_{t \in I} |g_1(t) - g_2(t)|$$

$$= Mh \|g_1 - g_2\| \qquad \forall x \in I.$$

Consequently, $\|Tg_1 - Tg_2\| \leq Mh \|g_1 - g_2\|$. ┘

Hence, $\exists^1 g \in X \subset C(I) : g(x) = y_0 + \int_{x_0}^{x} f(t, g(t))\,dt$

$(\forall x \in I)$.

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Remark

$g \in X \subset C(I)$

$$(Tg)(x) = y_0 + \int_{x_0}^{x} f(t, g(t))\,dt \quad \forall x \in I$$

- Picard iteration

$g_0 \in X \ (\subset C(I))$: given

$g_1 = T g_0$

$$g_1(x) = y_0 + \int_{x_0}^{x} f(t, g_0(t))\,dt \quad \forall x \in I$$

$g_2 = T g_1 = T^2 g_0$

$$g_2(x) = y_0 + \int_{x_0}^{x} f(t, g_1(t))\,dt \quad \forall x \in I$$

$\vdots$

- Mann iteration

Application: differential equation

1. Prove the Rolle's theorem.

2. Prove the mean value theorem.

3. Using the mean value theorem, prove the following:
Let I be a compact subset of $\mathbb{R}$. Let $f : I \to \mathbb{R}$ be a C^1 function, that is, f' is continuous on its domain I. Then, f is a Lipschitz function, that is, there is a real number $M \geq 0$ such that

$$|f(x) - f(y)| \leq M|x - y|$$

for all $x, y \in I$.

4. Show the following:
Let S be a metric space, let $f_n : S \to \mathbb{R}$ be continuous functions ($n \in \mathbb{N}$), and let $f : S \to \mathbb{R}$. Assume that $\{f_n\}$ converges to f uniformly, in other words,

$$\sup_{t \in S} |f_n(t) - f(t)| \to 0 \quad \text{as } n \to \infty.$$

Then, f is also continuous.

5. Prove that the following two statements are equivalent:

$$\begin{aligned} &(1) \quad y'(x) = f(x, y(x)) \quad \text{for all } x. \\ & \quad\ y(x_0) = y_0 \\ &(2) \quad y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt \quad \text{for all } x. \end{aligned}$$

6. Prove the main theorem of this section.