Sets and set inclusion (2)

Def.

A_μ : set $(\mu \in M)$

- $\bigcup_{\mu \in M} A_\mu = \{x \mid \exists \mu \in M : x \in A_\mu\}$
- $\bigcap_{\mu \in M} A_\mu = \{x \mid \forall \mu \in M, x \in A_\mu\}$

- $M = \{1, 2\}$

  Then, $\bigcup_{\mu \in M} A_\mu = A_1 \cup A_2$

  $\bigcap_{\mu \in M} A_\mu = A_1 \cap A_2$

- $M = \mathbb{N}$

  Then, $\bigcup_{\mu \in M} A_\mu = \bigcup_{n \in \mathbb{N}} A_n$

  $\bigcap_{\mu \in M} A_\mu = \bigcap_{n \in \mathbb{N}} A_n$

ex

$A_\alpha = \mathbb{R} \setminus \{\alpha\}$

Then, $x \in A_\alpha \iff \begin{cases} x \in \mathbb{R} \\ x \neq \alpha \end{cases}$.

Let $M = [0,1]$.

Then,

- $$\bigcup_{\alpha \in [0,1]} A_\alpha = \{x \in \mathbb{R} \mid \exists \alpha \in [0,1] : x \in A_\alpha\}$$
$$= \{x \in \mathbb{R} \mid \exists \alpha \in [0,1] : x \neq \alpha\}$$
$$= \mathbb{R}$$

- $$\bigcap_{\alpha \in [0,1]} A_\alpha = \{x \in \mathbb{R} \mid \forall \alpha \in [0,1],\ x \in A_\alpha\}$$
$$= \{x \in \mathbb{R} \mid \forall \alpha \in [0,1],\ x \neq \alpha\}$$
$$= [0,1]^c$$
$$= (-\infty, 0) \cup (1, \infty)$$

$X \neq \phi$

$A_\mu \subset X \quad (\mu \in M)$

Consider the case $M = \phi$.

$$\boxed{\bigcup_{\mu \in \phi} A_\mu = \phi}$$

<u>Proof</u>

Note that

$$x \in \bigcup_{\mu \in M} A_\mu$$

$$\Longleftrightarrow \exists \mu \in M : x \in A_\mu$$

When $M = \phi$, it is impossible.

Therefore, $\bigcup_{\mu \in \phi} A_\mu = \phi$.

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$$\boxed{\bullet \ \bigcap_{\mu \in \phi} A_\mu = X}$$

<u>Proof</u>

Note that

$$x \in \bigcap_{\mu \in M} A_\mu$$

$$\Longleftrightarrow \forall_{\mu \in M},\ x \in A_\mu .$$

When $M = \phi$, any element $x \in X$ satisfies this condition.

Hence, $\bigcap_{\mu \in \phi} A_\mu = X$.

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$$\boxed{\begin{array}{l} A_\mu \subset B \ (\forall \mu \in M) \\ \Longleftrightarrow \bigcup_{\mu \in M} A_\mu \subset B \end{array}}$$

Proof

($\Rightarrow$) Let $x \in \bigcup_{\mu \in M} A_\mu$.

i.e. $\exists \mu \in M : x \in A_\mu$.

As $A_\mu \subset B$, we have $x \in B$.

($\Leftarrow$) Let $\mu \in M$.

Choose $x \in A_\mu$ arbitrarily.

As $A_\mu \subset \bigcup_{\nu \in M} A_\nu \subset B$, we obtain $x \in B$.

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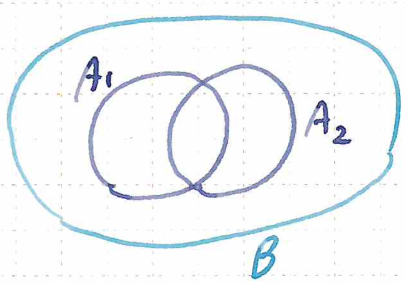
Special case

$$\boxed{\begin{array}{l} A_1 \subset B,\ A_2 \subset B \\ \Longleftrightarrow A_1 \cup A_2 \subset B \end{array}}$$



$$\boxed{\begin{gathered} A \subset B_\mu \quad (\forall \mu \in M) \\ \Longleftrightarrow A \subset \bigcap_{\mu \in M} B_\mu \end{gathered}}$$

<u>Proof</u>

($\Rightarrow$) Let $x \in A$.

We prove that $\underline{x \in \bigcap_{\mu \in M} B_\mu}$.

i.e $\forall \mu \in M$, $x \in B_\mu$.

Let $\mu \in M$.

From the hypothesis, we have

$x \in A \subset B_\mu$.

($\Leftarrow$) Let $\mu \in M$.

Choose $x \in A$ arbitrarily.

Our goal is to prove that $\underline{x \in B_\mu}$.

As $x \in A \subset \bigcap_{\nu \in M} B_\nu \subset B_\mu$, we have

$x \in B_\mu$.

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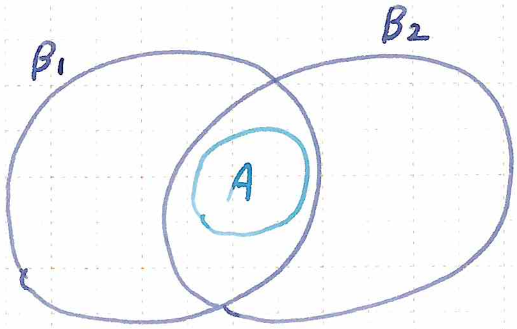
<u>Special case</u>

$$\boxed{\begin{gathered} A \subset B_1, \ A \subset B_2 \\ \Longleftrightarrow A \subset B_1 \cap B_2 \end{gathered}}$$



$$\boxed{A \cap \left(\bigcup_{\mu \in M} B_\mu \right) = \bigcup_{\mu \in M} (A \cap B_\mu)}$$ 分配法則

Proof

$x \in A \cap \left(\bigcup_{\mu \in M} B_\mu \right)$

$\Leftrightarrow x \in A$ and $x \in \bigcup_{\mu \in M} B_\mu$.

$\Leftrightarrow x \in A$ and $\exists \mu \in M ; x \in B_\mu$.

$\Leftrightarrow \exists \mu \in M : x \in A \cap B_\mu$

$\Leftrightarrow x \in \bigcup_{\mu \in M} (A \cap B_\mu)$.

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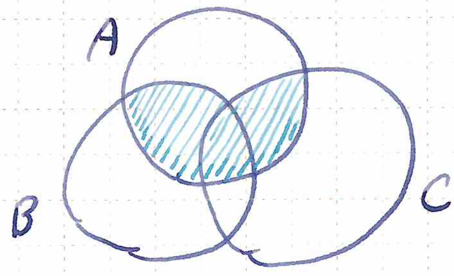
Special case

$$\boxed{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$$



$$\boxed{A \cup \left(\bigcap_{\mu \in M} B_\mu \right) = \bigcap_{\mu \in M} (A \cup B_\mu)}$$

<u>Proof</u>

$x \in \bigcap_{\mu \in M} (A \cup B_\mu)$

$\iff \forall \mu \in M,\ x \in A \cup B_\mu$

$\iff \forall \mu \in M,\ x \in A$ or $x \in B_\mu$ 注意!

$\iff x \in A$ or $x \in B_\mu\ (\forall \mu \in M)$

$\iff x \in A$ or $x \in \bigcap_{\mu \in M} B_\mu$

$\iff x \in A \cup \left(\bigcap_{\mu \in M} B_\mu \right).$

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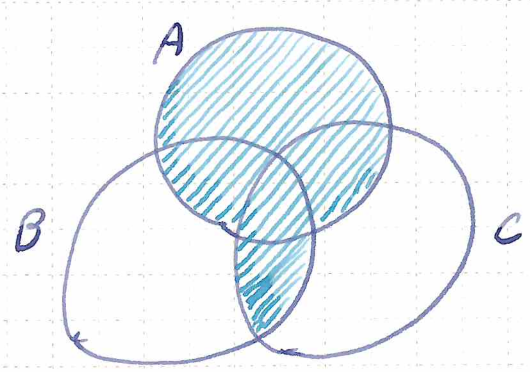
<u>Special case</u>

$$\boxed{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$$



ex

$A = \{a, b, c\}$

$B = \{b, c, d\}$

$C = \{d, e\}$

$$\boxed{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$$

check

LHS

- $B \cup C = \{b, c, d, e\}$

$\therefore A \cap (B \cup C)$

$= \{b, c\}$.

RHS

- $A \cap B = \{b, c\}$
- $A \cap C = \phi$

$\therefore (A \cap B) \cup (A \cap C) = \{b, c\}$

$\therefore$ LHS = RHS.

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$$\boxed{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$$

check

LHS

- $B \cap C = \{d\}$

$\therefore A \cup (B \cap C) = \underline{\{a, b, c, d\}}$

RHS

- $A \cup B = \{a, b, c, d\}$
- $A \cup C = \{a, b, c, d, e\}$

$\therefore (A \cup B) \cap (A \cup C)$

$= \underline{\{a, b, c, d\}}$

$\therefore LHS = RHS.$

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$$\boxed{\left(\bigcap_{\mu \in M} A_\mu\right)^c = \bigcup_{\mu \in M} A_\mu^c}$$

ド・モルガンの法則

Proof

$x \in \left(\bigcap_\mu A_\mu\right)^c$

$\iff x \notin \bigcap_\mu A_\mu$

$\iff \neg\left[\forall \mu \in M,\ x \in A_\mu\right]$

$\iff \exists \mu \in M ;\ x \notin A_\mu$

$\iff \exists \mu \in M ;\ x \in A_\mu^c$

$\iff x \in \bigcup_\mu A_\mu^c .$

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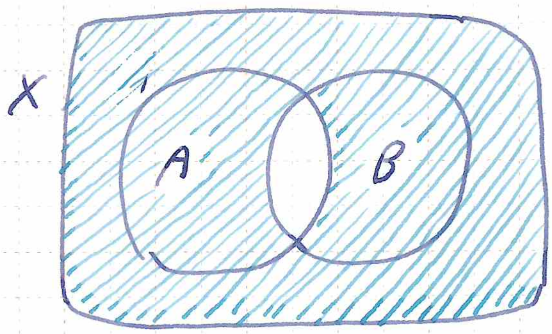
Special case

$$\boxed{(A \cap B)^c = A^c \cup B^c}$$



$$\boxed{\left(\bigcup_{\mu\in M} A_\mu\right)^c = \bigcap_{\mu\in M} A_\mu^c}$$

Proof

$x \in \left(\bigcup_\mu A_\mu\right)^c$

$\iff x \notin \bigcup_\mu A_\mu$

$\iff \neg\left[\exists \mu\in M;\ x\in A_\mu\right]$

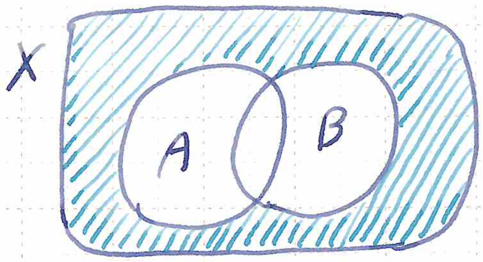
$\iff \forall \mu\in M,\ x\notin A_\mu$

$\iff \forall \mu\in M,\ x\in A_\mu^c$

$\iff x\in \bigcap_\mu A_\mu^c$. //

Special case

$$\boxed{(A\cup B)^c = A^c\cap B^c}$$



ex

$X = \{a, b, c, d, e\}$

$A = \{a, b, c\}$

$B = \{c, d\}$

$$\boxed{(A \cap B)^c = A^c \cup B^c}$$

check

LHS

- $A \cap B = \{c\}$

$\therefore (A \cap B)^c = \underline{\{a, b, d, e\}}$

RHS

- $A^c = \{d, e\}$
- $B^c = \{a, b, e\}$

$\therefore A^c \cup B^c = \underline{\{a, b, d, e\}}$

$\therefore LHS = RHS.$

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$$\boxed{(A \cup B)^c = A^c \cap B^c}$$

check

LHS

- $A \cup B = \{a, b, c, d\}$

$\therefore (A \cup B)^c = \{e\}$

RHS

$A^c \cap B^c = \{e\}$

$\therefore$ LHS = RHS. //

Sets and set inclusion (2)

1. A_μ $(\mu \in M)$を集合族とする. $M = \emptyset$のとき,

$$\bigcup_{\mu \in M} A_n = \emptyset$$

と考えるのが自然である理由を説明せよ.

2. Xを空でない集合, A_μ $(\mu \in M)$をその部分集合族とする. $M = \emptyset$のとき,

$$\bigcap_{\mu \in M} A_n = X$$

と考えるのが自然である理由を説明せよ.

3.以下を証明せよ. また, 具体例を自分で作って説明せよ.
(1) $A_1, A_2 \subset B \Leftrightarrow A_1 \cup A_2 \subset B$
(2) $A \subset B_1, B_2 \Leftrightarrow A \subset B_1 \cap B_2$

4. 以下を証明せよ. また, ベン図を描いてみよ.
(1) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(2) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

5. 集合

$$A = \{a, b, c\},$$
$$B = \{c, d\},$$
$$C = \{a, e\}$$

とする. 問題4の(1)(2)について, 左辺と右辺のそれぞれがあらわす集合を求め, 両辺が等しいことを確認せよ.

6. A, BをXの部分集合とする. 以下を証明し, 具体例を自分で考えて説明せよ.
(1) $(A \cap B)^C = A^C \cup B^C$
(2) $(A \cup B)^C = A^C \cap B^C$

7. 集合Xの3つの部分集合A, B, Cについて, ド・モルガンの法則を述べベン図を描いて説明せよ.