# Equivalence relations

**Def**

$X$ set, $\neq \phi$

$\rho$ : binary relation on $X$

$\Leftrightarrow$ $^\forall x, y \in X$, $x \rho y$ or $x \not\rho y$

**ex**

$X = \mathbb{R}$

$x \rho y \underset{\text{def.}}{\Longleftrightarrow} y = x^2$

Then, $3 \rho 9$, $3 \not\rho 10$,

$2 \rho 4$, $4 \not\rho 2$

**ex**

$X$ set, $\neq \phi$

$\subset$ is a binary relation on $2^X$.

i.e. $^\forall A, B \in 2^X$, $A \subset B$ or $A \not\subset B$.

$X$ set, $\neq \emptyset$

$\rho$: binary relation on $X$

**Def.**

$\rho$: <u>equivalence relation</u> on $X$

$\Leftrightarrow$ (E1) ${}^\forall x \in X,\ x \rho x$ (reflexivity)

(E2) $x \rho y \Rightarrow y \rho x$ (symmetry)

(E3) $x \rho y,\ y \rho z \Rightarrow x \rho z$

(transitivity)

The symbol "$\sim$" is often need

when a binary relation is equivalence.

<u>ex</u>

$X = \mathbb{R}$

Then, "$=$" is an equivalence relation on $\mathbb{R}$.

<u>ex</u>

$X$ set, $\neq \phi$

Then, "$=$" is an equivalence relation on $2^X$.

<u>ex</u>

$X = \mathbb{R}$

$x \rho y \iff y = x^2$

Then, $\rho$ is <u>not</u> an equivalence relation on $\mathbb{R}$.

$X \neq \phi$

$\rho$ : binary relation on $X$

$\rho$ satisfies transitivity

i.e. $u \rho v,\ v \rho w \Rightarrow u \rho w$ —— (*)

$x \rho y,\ y \rho z,\ z \rho w$

$\Rightarrow x \rho w$

Proof

As $x \rho y$ and $y \rho z$, we have from (*) that

$$x \rho z.$$

As $x \rho z$ and $z \rho w$ hold true, from (*),

$$x \rho w.$$

//

ex

$\mathbb{Z}$

$x \rho y \overset{\text{def.}}{\Longleftrightarrow} x - y = 7n$ for some $n \in \mathbb{Z}$

$\Rightarrow \rho$ : equivalence relation on $\mathbb{Z}$.

Proof

(E1) $x \rho x$

i.e. $\forall x \in \mathbb{Z}, \exists n \in \mathbb{Z} : x - x = 7n$.

OK. $\lrcorner$

(E2) $x \rho y \Rightarrow y \rho x$

Assume that $x \rho y$.

i.e. $\exists m \in \mathbb{Z} : x - y = 7m$

We show that $y \rho x$.

i.e. $\exists n \in \mathbb{Z} : y - x = 7n$

Let $n = -m \in \mathbb{Z}$.

Then, $y - x = -(x - y)$

$= -7m$

$= 7(-m)$

$= 7n.$ $\lrcorner$

(E3) $x \rho y,\ y \rho z \Rightarrow x \rho z$

Assume that $x \rho y$ and $y \rho z$.

i.e. $\exists l \in \mathbb{Z} : x - y = 7l$

$\exists m \in \mathbb{Z} : y - z = 7m$

We demonstrate that $x \rho z$.

i.e. $\exists n \in \mathbb{Z} : x - z = 7n$

It holds that

$$
\begin{aligned}
& x - z \\
&= (x-y) + (y-z) \\
&= 7l + 7m \\
&= 7(l+m).
\end{aligned}
$$

Letting $n \equiv l + m \in \mathbb{Z}$, we obtain

$x - z = 7n$, where $n \in \mathbb{Z}$.

This shows that $x \rho z$.

//

**Def.**

$X$ set, $\neq \phi$

$\sim$ : equivalence relation on $X$

$a \in X$

$C_a \equiv \{x \in X \mid x \sim a\}$

<u>equivalence class</u> of $a$

同値類

<u>ex</u>

$\mathbb{Z}$

$x \sim y \iff x - y = 7n$ for some $n \in \mathbb{Z}$

$1 \in \mathbb{Z}$

Then,

$$C_1 = \{x \in \mathbb{Z} \mid x \sim 1\}$$
$$= \{x \in \mathbb{Z} \mid x - 1 = 7n \text{ for some } n \in \mathbb{Z}\}$$
$$= \{\cdots, -13, -6, 1, 8, 15, \cdots\}$$

<u>ex</u>

$C^1(a,b) = \{f : (a,b) \to \mathbb{R} \mid f' : \text{continuous}\}$

$f \sim g \iff f' = g'$

Let $h(x) = x^2 \in C^1(a,b)$.

Then, $C_h = \{f \in C^1(a,b) \mid f \sim h\}$

$$= \{x^2 + C \mid C \in \mathbb{R}\}$$

Th

$X$ set, $\neq \phi$

$\sim$ : equivalence relation on $X$

$C_a \equiv \{x \in X \mid x \sim a\}$, where $a \in X$

$\Rightarrow$ (1) $a \in C_a \quad \forall a \in X$

(2) $a \sim b \Leftrightarrow C_a = C_b$

(3) $a \nsim b \Leftrightarrow C_a \cap C_b = \phi$

Proof

(1) $\underline{a \in C_a}$ i.e. $a \sim a$.

OK

(2) ($\Rightarrow$) ($\subset$) Let $x \in C_a$. i.e. $x \sim a$ — (*)

We show that $\underline{x \in C_b}$. i.e. $x \sim b$.

From the hypothesis, $a \sim b$. — (**)

From (*) and (**), $x \sim b$.

($\supset$) Let $x \in C_b$. i.e. $x \sim b$. — (*3)

We show that $\underline{x \in C_a}$. i.e. $x \sim a$.

From the hypothesis, $a \sim b$.

Thus, $b \sim a$. — (*4)

From (*3) and (*4), $x \sim a$.

($\Leftarrow$) From (1), $a \in C_a = C_b$.

Therefore, $a \sim b$.

(3) We show that $a \sim b \Leftrightarrow C_a \cap C_b \neq \phi$.

($\Rightarrow$) From (1), $a \in C_a$.

As $a \sim b$ is assumed, $a \in C_b$.

Therefore, $a \in C_a \cap C_b$.

$\therefore C_a \cap C_b \neq \phi$. ⌟

($\Leftarrow$) Let $x \in C_a \cap C_b$.

Then, $\begin{cases} x \sim a & \therefore a \sim x. \\ x \sim b. & \end{cases}$

Hence, we obtain $a \sim b$. //

$\sim$ : equivalence relation on $X$

(E1) $x \sim x$

(E2) $x \sim y \Rightarrow y \sim x$

(E3) $x \sim y,\ y \sim z \Rightarrow x \sim z$

Remark

$\{C_x\}_{x \in X}$ : partition of $X$

$$\Leftrightarrow \left| \begin{array}{l} \text{(i) } \forall x \in X,\ C_x \neq \emptyset \\ \text{(ii) } x \neq x' \Rightarrow C_x \cap C_{x'} = \emptyset \\ \text{(iii) } X = \bigcup_{x \in X} C_x \end{array} \right.$$

# Equivalence relations

1. 二項関係と同値関係の定義を述べよ.

2. 同値関係の例と，二項関係だが同値関係にはならない例を挙げよ.

3. 空ではない2つの集合$X, Y$に対して

$$X \sim Y$$
$$\Leftrightarrow \text{ある1対1かつ上への写像} f : X \to Y \text{が存在する.}$$

と定めると，この関係は同値関係の3条件を満たすことを示せ.

4. 空ではない集合$X$上に同値関係$\sim$が定まっている. 集合$X$の要素$a \in X$に対して

$$C_a = \{x \in X : x \sim a\}$$

として$a$の同値類を定義すると，次の(1)-(3)が成り立つ. このことを示せ.

(1) $a \in C_a$
(2) $a \sim b \Leftrightarrow C_a = C_b$
(3) $a \nsim b \Leftrightarrow C_a \cap C_b = \emptyset$