Order relation

$X \neq \phi$

$\leqq$: binary relation defined on X

i.e. $\forall x, y \in X$, $x \leqq y$ or $x \nleqq y$

Def

$(X, \leqq)$ (partially) ordered set

$\Leftrightarrow$ (01) (reflexivity)

$$x \leqq x$$

(02) (anti-symmetry)

$$x \leqq y,\ y \leqq x \Rightarrow x = y$$

(03) (transitivity)

$$x \leqq y,\ y \leqq z \Rightarrow x \leqq z$$

(02) $\Leftrightarrow$ $x \leqq y,\ x \neq y \Rightarrow y \nleqq x$

ex

$(\mathbb{R}, \leqq)$ ordered set

ex

$\mathbb{R}^2$

$(x, y) \leqq (u, v)$

$\Longleftrightarrow \begin{cases} x \leqq u & \text{and} \\ y \leqq v \end{cases}$

Then, $(\mathbb{R}^2, \leqq)$ is an ordered set.

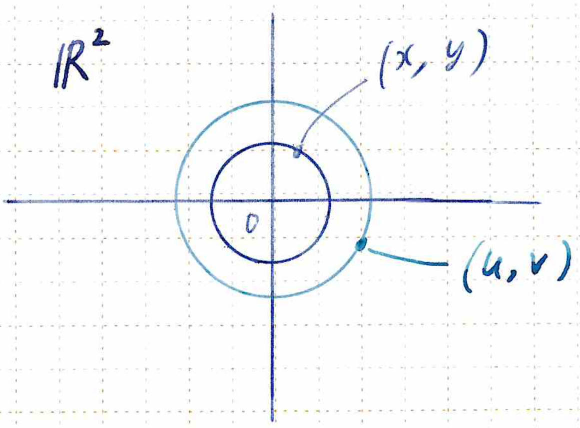
ex

$\mathbb{R}^2$

$(x, y) \leqq (u, v)$

$\Longleftrightarrow \sqrt{x^2+y^2} \leqq \sqrt{u^2+v^2}$

Then, $(\mathbb{R}^2, \leqq)$ is not an ordered set.



<u>ex</u>

$X \neq \phi$

$A, B \in 2^X$

$A \leq B \iff A \subset B$

Then, $(2^X, \leq)$ is an ordered set.

$X \neq \emptyset$

$\leqq$: binary relation on X

i.e. $\forall x, y \in X$, $x \leqq y$ or $x \nleqq y$

<u>Def</u>

$(X, \leqq)$ <u>linearly (totally) ordered set</u>

$\Leftrightarrow$ (I) $(X, \leqq)$ ordered set

(II) (completeness)

$x \leqq y$ or $y \leqq x$

X totally ordered set

$x \nleqq y$

$\Rightarrow y \leqq x$

※ (II) completeness

$\Rightarrow \forall x \in X, \ x \leqq x$

(reflexivity)

<u>ex</u>

$(\mathbb{R}, \leqq)$: linearly ordered set

<u>ex</u>

$\mathbb{R}^2$

$(x_1, y_1) \leqq (x_2, y_2)$

$\Longleftrightarrow \begin{cases} x_1 \leqq x_2 \text{ and} \\ y_1 \leqq y_2 \end{cases}$

Then, $(\mathbb{R}^2, \leqq)$ is an ordered set.

However, it is <u>not</u> a totally ordered set.

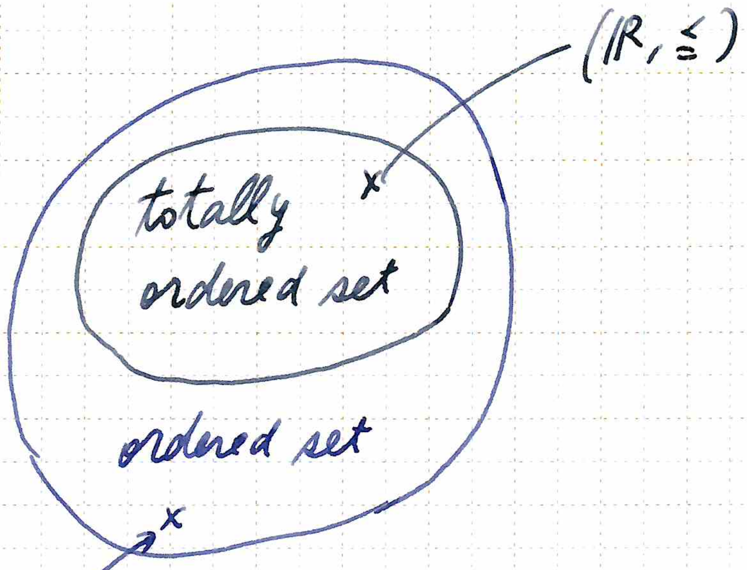
<u>ex</u>

$X \neq \phi$

$A \leqq B \Longleftrightarrow A \subset B$

Then, $(2^X, \leqq)$ is an ordered set, but

it is <u>not</u> a totally ordered set.



- $\mathbb{R}^2$ with

  $$(x,y) \le (u,v) \iff \begin{cases} x \le u \\ y \le v \end{cases}$$

- $X \neq \emptyset$

  2^X with

  $$A \le B \iff A \subset B$$

Def.

$(X, \leqq)$ ordered set

$A \subset X$

A : bounded from above

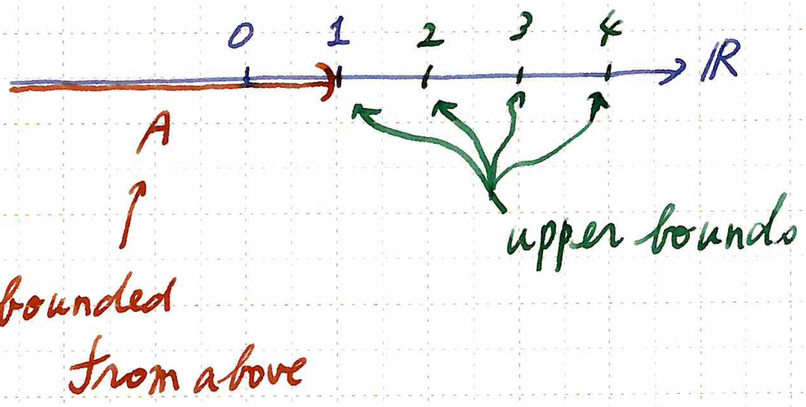
$\Longleftrightarrow \exists a \in X ; \forall x \in A, x \leqq a$

a : an upper bound of A

ex

$X = \mathbb{R}$

$A = (-\infty, 1)$



Def

$(X, \leq)$ ordered set

$A \subset X$

A : <u>bounded</u>

$\Longleftrightarrow$ A is bounded from above and bounded from below

A : bounded

$$\Longleftrightarrow \begin{cases} \exists M \in X : \forall x \in A,\ x \leq M & \text{and} \\ \exists L \in X : \forall x \in A,\ L \leq x \end{cases}$$

ex

$[0,1] \subset \mathbb{R}$ bdd

Indeed, $\exists_{2 \in \mathbb{R}} : \forall_{x \in [0,1]},\ x \leq 2$

$\exists_{-1 \in \mathbb{R}} : \forall_{x \in [0,1]},\ -1 \leq x.$

ex

$X \neq \emptyset$

$(2^X, \subset)$: ordered set

$A, B \in 2^X$

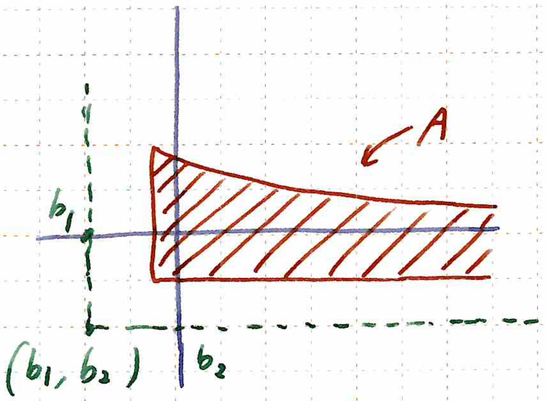
Then, $\{A, B\} \subset 2^X$ is bdd.

Indeed,

$A \cup B,\ X$: upper bounds

$A \cap B,\ \emptyset$: lower bounds.

ex



- A is founded from below

i.e. $\exists (b_1, b_2) \in \mathbb{R}^2$:

$\forall (x, y) \in A,\ (b_1, b_2) \leq (x, y)$

i.e. $\begin{cases} b_1 \leq x \text{ and} \\ b_2 \leq y \end{cases}$

- A is not founded above.

i.e $\forall (a_1, a_2) \in \mathbb{R}^2$,

$\exists (x, y) \in A : (x, y) \not\leq (a_1, a_2)$

i.e. $\begin{cases} x \not\leq a_1 \text{ or} \\ y \not\leq a_2 \end{cases}$

Order relations

1. 二項関係，順序関係，全順序関係の定義を述べよ．また，順序関係だが全順序関係ではない例を2つ挙げ（レジュメの例で構わない），なぜそうなのかを説明せよ．

2. 実数の集合$\mathbb{R}$上に二項関係$\leq$を

$$x \leq y \Leftrightarrow |x| \leq |y|$$

と定める．これは順序関係になるだろうか？ ならないとすると，3条件のうちどれが満たされないか？

3. 順序集合$(X, \leq)$の空ではない部分集合Aがあるとする．これが上に有界，下に有界であることの定義を述べよ．また，上界と下界の定義を述べよ．

4. $\mathbb{R}^2$にレジュメで述べた順序関係が定まっているとする．
(1) $\mathbb{R}^2$の部分集合Aを

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

と定める．Aの上界と下界を2つずつ答えよ．
(2) 上に有界だが下には有界ではない$\mathbb{R}^2$の部分集合の例を挙げよ．