Supremum, maximum

Def

$(X, \leqq)$ ordered set

$A \subset X, \neq \phi$

$u \in X$ least upper bound of A

supremum

最小上界
上限

$\Longleftrightarrow \begin{cases} ① \ {}^\forall x \in A,\ x \leqq u & (上界) \\ ② \ {}^\forall x \in A,\ x \leqq a \Rightarrow u \leqq a & (最小a) \end{cases}$

※

$(X, \leqq)$ ordered set

(01) $x \leqq x$

(02) $x \leqq y,\ y \leqq x \Rightarrow x = y$

(03) $x \leqq y,\ y \leqq z \Rightarrow x \leqq z$

ex

$(\mathbb{R}, \leq)$ ordered set

$(0,1) \subset \mathbb{R}$

Then, 1 is the least upper bound of $(0,1)$.

i.e. $\sup(0,1) = 1$.

ex

$A = \{3, 3.1, 3.14, \cdots\} \subset \mathbb{R}$

Then, $\pi = \sup A$.

ex

$\{1, 3\} \subset \mathbb{R}$

Then, $3 = \sup\{1, 3\}$.

ex.

$X \neq \phi$

$(2^X, \subset)$ ordered set

$A, B \in 2^X$ (i.e. $A, B \subset X$)

$\{A, B\} \subset 2^X$

Then, $A \cup B$ is the least upper bound of $\{A, B\}$.

i.e. $\begin{cases} ① \ A \subset A \cup B, \quad B \subset A \cup B \\ ② \ A \subset C, \ B \subset C \Rightarrow A \cup B \subset C \end{cases}$

Proof

① holds obviously.

②: Assume that $A \subset C$ and $B \subset C$. —(*)

We show that $A \cup B \subset C$.

Let $x \in A \cup B$.

i.e. $x \in A$ or $x \in B$.

From (*), we obtain $x \in C$.

//

ex

$\mathbb{R}^2$

$A = \{(1,0), (0,1)\} \subset \mathbb{R}^2$

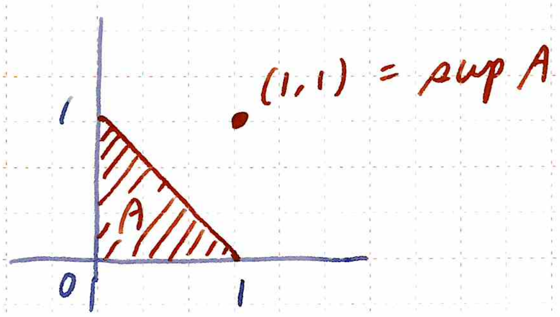
Then, $\underline{\sup A = (1,1)}$.

i.e. ① $(1,0) \leq (1,1)$, $(0,1) \leq (1,1)$

② $(1,0) \leq (a,b)$, $(0,1) \leq (a,b)$ $\Longrightarrow (1,1) \leq (a,b)$

ex

$\mathbb{R}^2$



$(X, \leq)$ ordered set

$A \subset X, \neq \phi$

$u_1 = \sup A$

$u_2 = \sup A$

$\Longrightarrow u_1 = u_2$

Proof

As $u_1 = \sup A$,

$$\begin{cases} ① \ \forall x \in A, \ x \leq u_1 \\ ② \ \forall x \in A, \ x \leq a \Rightarrow u_1 \leq a \end{cases}$$

As $u_2 = \sup A$,

$$\begin{cases} (1) \ \forall x \in A, \ x \leq u_2 \\ (2) \ \forall x \in A, \ x \leq a \Rightarrow u_2 \leq a \end{cases}$$

From ① & (2), $u_2 \leq u_1$.

From (1) & ②, $u_1 \leq u_2$.

Consequently, $u_1 = u_2$. ← (02) anti-symmetry $x \leq y, \ y \leq x \Rightarrow x = y$

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$(X, \leq)$ ordered set

(01) $x \leq x$

(02) $x \leq y, \ y \leq x \Rightarrow x = y$

(03) $x \leq y, \ y \leq z \Rightarrow x \leq z$

Def.

$(X, \leqq)$ ordered set

$A \subset X, \neq \emptyset$

$a \in X$ is a <u>maximum</u> of A 最大要素

$\Longleftrightarrow$ ① $a \in A$

② ${}^{\forall} x \in A,\ x \leqq a$

$a \in X$ is <u>not</u> a maximum of A

$\Longleftrightarrow \begin{cases} (1)\ a \notin A \text{ or} \\ (2)\ {}^{\exists} x \in A : x \nleqq a \end{cases}$

${}^{\exists} a = \max A$

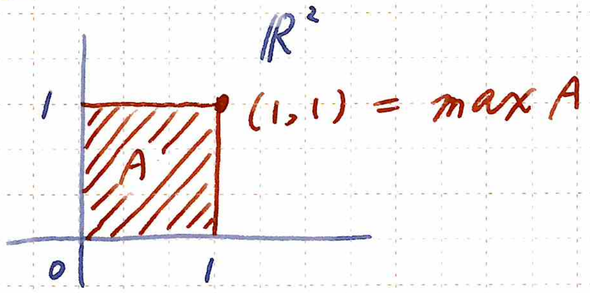
$\Rightarrow A$: bdd above

ex

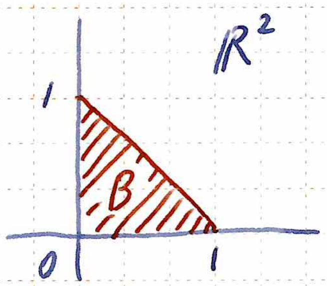
$(0,1) \subset \mathbb{R}$

Then, there is no maximum of $(0,1)$.

ex



ex



There is not maximum of B.

- $(1,0)$ is not a maximum of B.

  Indeed,

  $\exists (0,1) \in B : (0,1) \not\leq (1,0)$

  i.e. $\left(\begin{array}{l} 0 \not\leq 1 \text{ or} \\ 1 \not\leq 0 \end{array} \right.$

$(X, \leq)$ ordered set

$A \subset X, \neq \emptyset$

$\alpha = \max A$

$\beta = \max A$

$\Rightarrow \alpha = \beta$

<u>Proof</u>

As $\alpha = \max A$, it holds that

$$\begin{cases} ① \ \alpha \in A; \\ ② \ \forall x \in A,\ x \leq \alpha. \end{cases}$$

As $\beta = \max A$,

$$\begin{cases} (1) \ \beta \in A; \\ (2) \ \forall x \in A,\ x \leq \beta. \end{cases}$$

From ① & (2), $\alpha \leq \beta$.

From (1) & ②, $\beta \leq \alpha$.

Therefore, $\alpha = \beta$.

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$(X, \leqq)$ ordered set

$A \subset X, \neq \phi$

$a = \max A$

i.e. $\begin{cases} (1)\ a \in A \\ (2)\ {}^\forall x \in A,\ x \leqq a \end{cases}$

$\Rightarrow a = \sup A$

i.e. $\begin{cases} ①\ {}^\forall x \in A,\ x \leqq a \\ ②\ {}^\forall x \in A,\ x \leqq \alpha \Rightarrow a \leqq \alpha \end{cases}$

Proof

As $(2) \Leftrightarrow ①$, it suffices to demonstrate that ② holds.

Assume that ${}^\forall x \in A,\ x \leqq \alpha$. — $(*)$

From (1), $a \in A$.

Thus, letting $x = a \in A$ in $(*)$, we obtain $a \leqq \alpha$.

We obtain ${}^\forall x \in A,\ x \leqq \alpha \Rightarrow a \leqq \alpha$.

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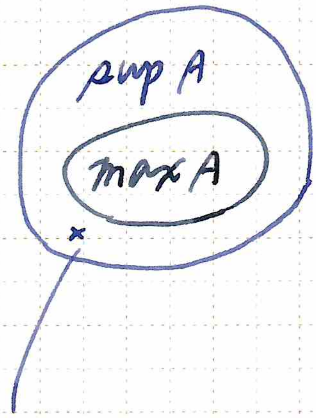

ex

$A=(0,1)\subset\mathbb{R}$

Then, • $1=\sup A$.

However, • $1 \neq \max A$.

ex

$X \neq \emptyset$

$(2^X, \subset)$ ordered set

$A, B \in 2^X$: $A \neq B$

• $\mathcal{A} = \{A, B\} \subset 2^X$

Then, $\sup \mathcal{A} = A \cup B$

There is not max $\mathcal{A}$.

• 2^X

$\sup 2^X = \max 2^X = X$.

$(X, \leqq)$ ordered set

$A \subset X$

$u = \sup A \in A$

i.e. $\begin{cases} ① \ \forall x \in A,\ x \leqq u \\ ② \ \forall x \in A,\ x \leqq a \Rightarrow u \leqq a \end{cases}$

$\Longrightarrow u = \max A$

i.e. $\begin{cases} (1) \ u \in A \\ (2) \ \forall x \in A,\ x \leqq u \end{cases}$

Proof

(1) directly follows from the hypothesis.

(2) is equivalent with ①.

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※ $\sup A$ が A の要素であれば、

それは A の最大元である。

Appendix

条件付き最大化問題

$X \neq \phi$

$f : X \to \mathbb{R}$

$A \subset X$

$$\boxed{\max_{x \in A} f(x)}$$

$$\boxed{\begin{aligned} &\max_{x} f(x) \\ &\quad \text{s.t. } x \in A \end{aligned}} \quad \text{—} (*)$$

s.t. = subject to

〜の制約の下で

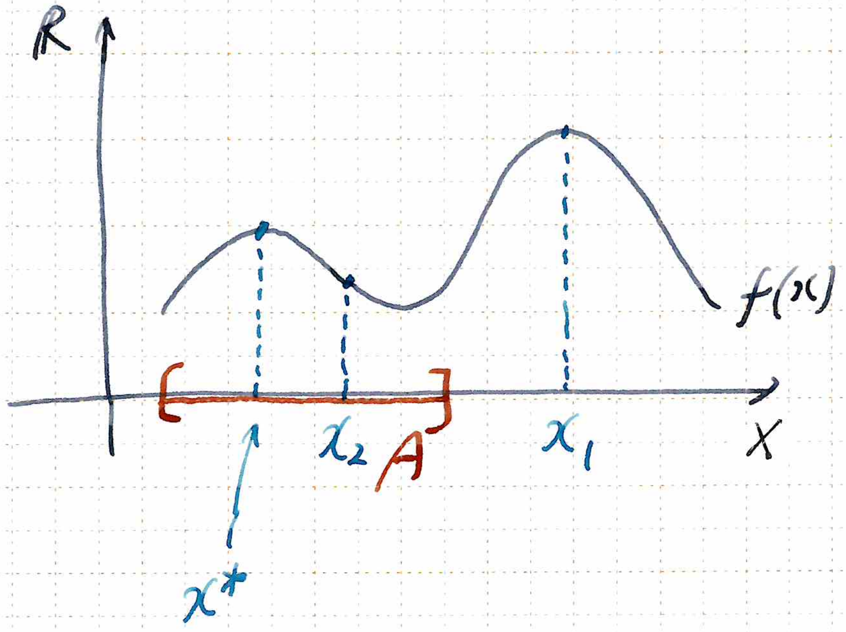
Def

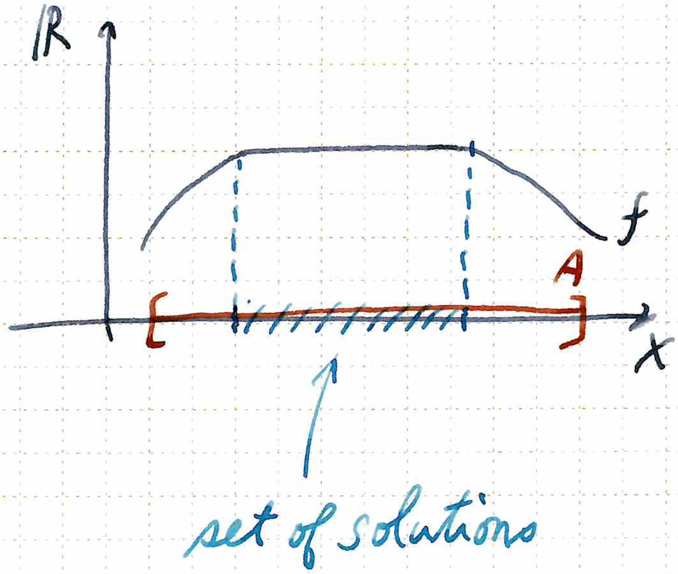
$x^* \in X$: solution to (*)

$\Longleftrightarrow \begin{cases} ① \ x^* \in A \\ ② \ {}^\forall x \in A,\ f(x) \leq f(x^*) \end{cases}$

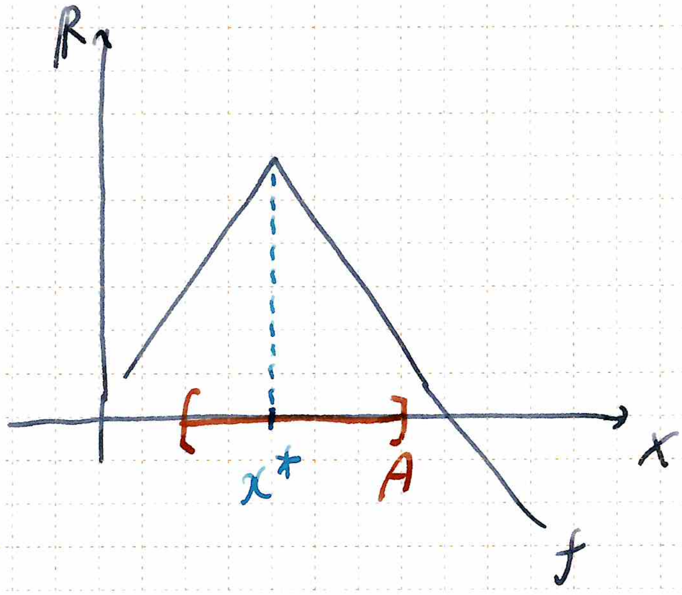
ex

$X = \mathbb{R}$

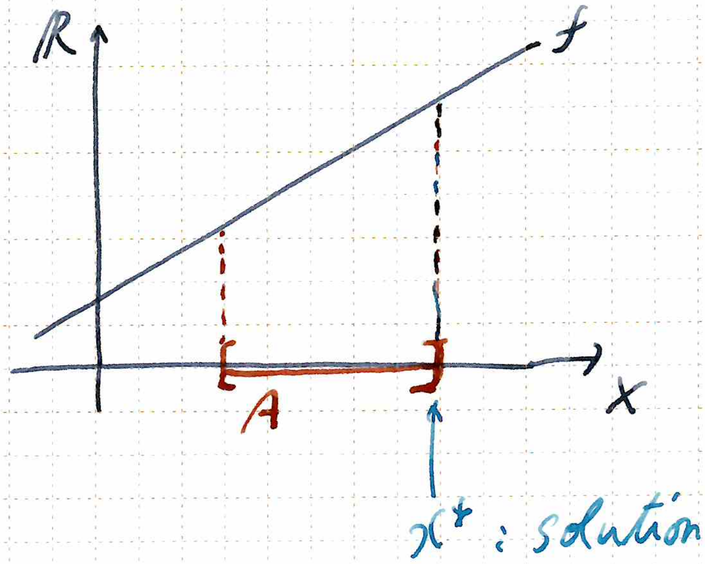


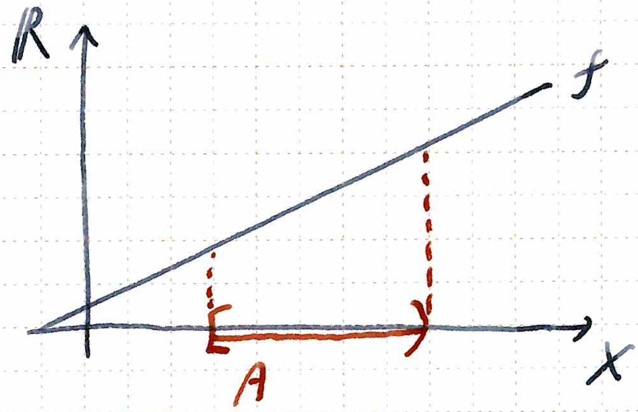


R
f
A
x
set of solutions



R
x
x^*
A
f





There is not a solution to (*).

Supremum, maximum

1. 順序集合$(X, \leq)$の空ではない部分集合Aがあるとする.
(1) 二項関係と順序集合の定義を復習せよ.
(2) Aの最小上界(supremum)の定義を述べよ.
(3) Aの最大下界(infimum)の定義を述べよ.
(4) Aの最大要素(maximum)の定義を述べよ.
(5) Aの最小要素(mimimum)の定義を述べよ.

2. $\mathbb{R}$の以下の部分集合について, 最小上界, 最大下界, 最大要素, 最小要素を答えよ(存在しない場合もある).
(1) $A = (0, 1]$ (2) $B = (-\infty, 1)$ (3) $C = \{1, 2, 3\}$

	sup	inf	max	min
A				
B				
C				

3. 集合A, Bについて, 二項関係を$A \leq B \Leftrightarrow A \subset B$と定めたとき, $A \cup B$が$\{A, B\}$の最小上界であることを確認せよ.

4. 集合$A = \{a, b, c\}, B = \{b, c, d, e\}$について, 最大下界を求めよ.

5. 順序集合$(X, \leq)$の空ではない部分集合Aについて, 最小上界と最大要素が一意であることを証明せよ.

6. 順序集合$(X, \leq)$の部分集合Aについて, Aの上限αがAの要素のとき, $\alpha = \max A$である. これを示せ.