Real numbers and sequences

Number system

- Natural numbers

  $\mathbb{N} = \{1, 2, 3, \cdots\}$

- Integers

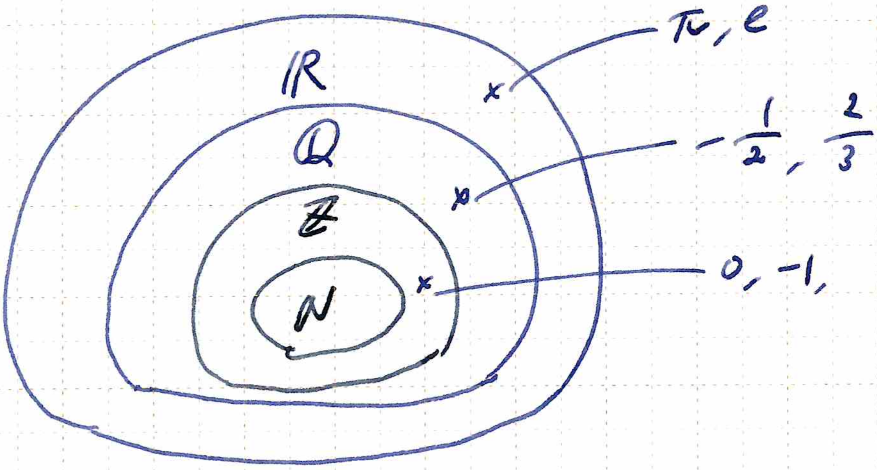
  $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$

- Rational numbers

  $\mathbb{Q} = \left\{ \frac{m}{n} \,\middle|\, m \in \mathbb{Z},\ n \in \mathbb{N} \right\}$

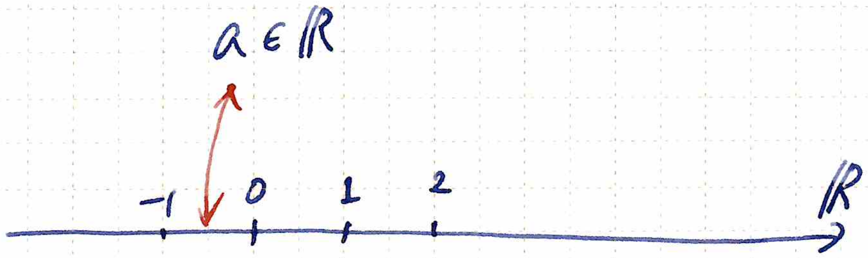
- Real numbers

  $\mathbb{R} = (-\infty, \infty)$ ※ $-\infty, \infty \notin \mathbb{R}$



$\mathbb{R}$ the set of real numbers

(I) real line



(II) $+, -, \times, \div$

(III) $x, y \in \mathbb{R}$

$|x-y|$ ⟵ the distance between two points x and y

$|x| = |x-0|$

↖ the distance between x and 0

(IV) $(\mathbb{R}, \leqq)$

(O1) $x \leqq x$

(O2) $x \leqq y,\ y \leqq x \Rightarrow x = y$

(O3) $x \leqq y,\ y \leqq z \Rightarrow x \leqq z$

(O4) $x \leqq y$ or $y \leqq x$

Fundamental properties of $|\cdot|$

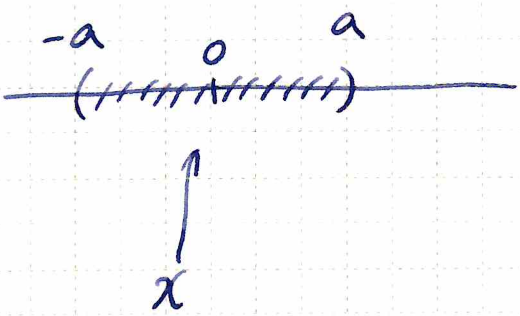
(A1) $|x| \geqq 0$; $|x| = 0 \iff x = 0$

(A2) $|\alpha x| = |\alpha| |x|$

(A3) $|x + y| \leqq |x| + |y|$

※ $|x| \leqq a$

$\iff -a \leqq x \leqq a$



$$\boxed{\begin{array}{l} x, y, z \in \mathbb{R} \\ \Rightarrow |x+y+z| \leq |x| + |y| + |z| \end{array}}$$

Proof.

It holds true that

$$\begin{aligned} &|x+y+z| \\ &= |(x+y)+z| \\ &\leq |x+y| + |z| && \text{(A3)} \\ &\leq (|x|+|y|) + |z| && \text{(A3)} \\ &= |x| + |y| + |z|. \end{aligned}$$

//

$$\boxed{|x-y| \leq |x| + |y|}$$

<u>Proof</u>

It follows that

$$
\begin{aligned}
|x-y| & \\
&= |x+(-y)| \\
&\leq |x| + |-y| && \text{(A3)} \\
&= |x| + |(-1)\cdot y| \\
&= |x| + |-1|\cdot|y| && \text{(A2)} \\
&= |x| + 1\cdot|y| \\
&= |x| + |y|.
\end{aligned}
$$

//

(A1) $|x| \geqq 0$; $|x| = 0 \Longleftrightarrow x = 0$

(A2) $|\alpha x| = |\alpha|\,|x|$

(A3) $|x+y| \leqq |x| + |y|$

$$\boxed{\big||x|-|y|\big| \leqq |x-y|}$$

Proof

Our goal is to prove that

$$\underbrace{-|x-y| \leqq}_{①} |x|-|y| \underbrace{\leqq |x-y|}_{②}.$$

②: It holds that

$$\begin{aligned} |x| &= |x-y+y| \\ &\leqq |x-y|+|y|. \end{aligned}$$

$$\therefore |x|-|y| \leqq |x-y|.$$

①: The following holds:

$$\begin{aligned} |y| &= |-y| \\ &= |x-y-x| \\ &\leqq |x-y|+|-x| \\ &= |x-y|+|x| \end{aligned}$$

$$\therefore -|x-y| \leqq |x|-|y|.$$

Hence, we obtain the desired result. //

Def

$$\boxed{S : \mathbb{N} \to \mathbb{R} \quad \underline{\textit{real sequence}}}$$

$$\left(S : \mathbb{N} \cup \{0\} \to \mathbb{R} \right)$$

$$\{a_n\}, \quad \{a_n\}_{n=1}^{\infty}, \quad \{a_n\}_{n \in \mathbb{N}}$$

$$\begin{array}{ccccccc}
 & & \mathbb{R} & \mathbb{R} & \mathbb{R} & & \\
 & & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & & \\
\{a_n\} = & \{ & a_1, & a_2, & a_3, & a_4, & \cdots \} \\
 & & \Big\uparrow & \Big\uparrow & \Big\uparrow & & \\
 & & 1 & 2 & 3 & & \\
 & & \rotatebox{90}{$\ni$} & \rotatebox{90}{$\ni$} & \rotatebox{90}{$\ni$} & & \\
 & & \mathbb{N} & \mathbb{N} & \mathbb{N} & &
\end{array}$$

Def

$$\boxed{\begin{array}{l} \{a_n\} = \{b_n\} \\ \iff \forall n \in \mathbb{N}, \ a_n = b_n \end{array}}$$

※ 数列は各項からなる(各項を要素とする)集合ともみなせる。

$$\{a_n\} = \{a_1, a_2, a_3, \cdots\}$$
$$= \{a_n \in \mathbb{R} \mid n \in \mathbb{N}\}$$

※しかし、数列と集合は区別すべき。

$$\{1, 0, 1, 0, \cdots\} \neq \{0, 1, 0, 1, \cdots\}$$

これらは数列としては別物。

しかし、集合とみなすと、共に

$$\{1, 0\}$$

に等しい。

※数列は集合($\mathbb{R}$の部分集合)ともみなせるので

$$\{a_n\} \subset \mathbb{R}$$

と書くこともある。

Def

$X \neq \emptyset$

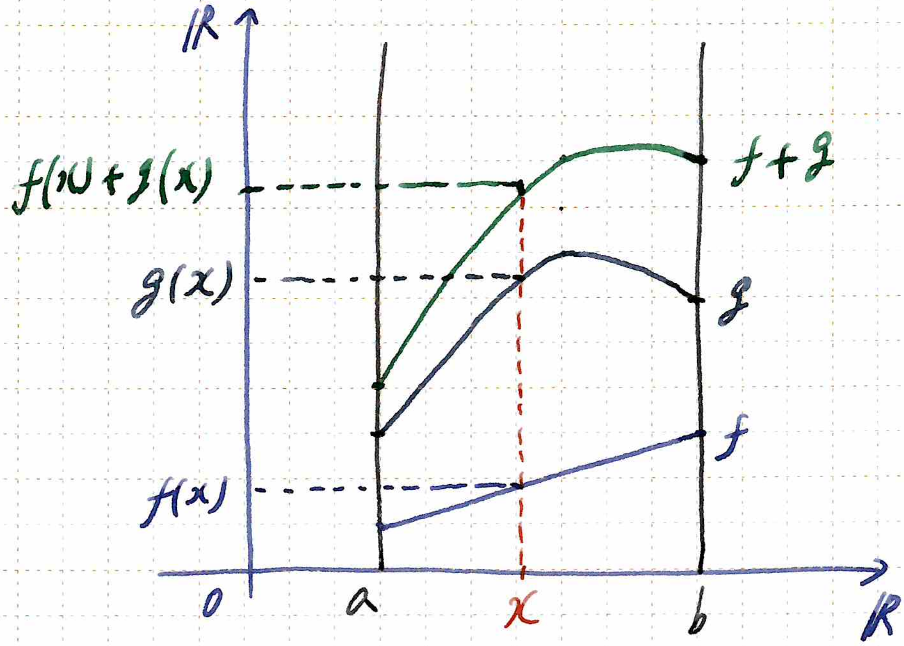
$f, g : X \to \mathbb{R}$

$\alpha \in \mathbb{R}$

- $(f+g)(x) := f(x) + g(x) \quad \forall x \in X$
- $(\alpha f)(x) := \alpha \cdot f(x) \quad \forall x \in X$
- $(f \cdot g)(x) := f(x) \cdot g(x) \quad \forall x \in X$
- $\dfrac{f}{g}(x) := \dfrac{f(x)}{g(x)} \quad \forall x \in X$

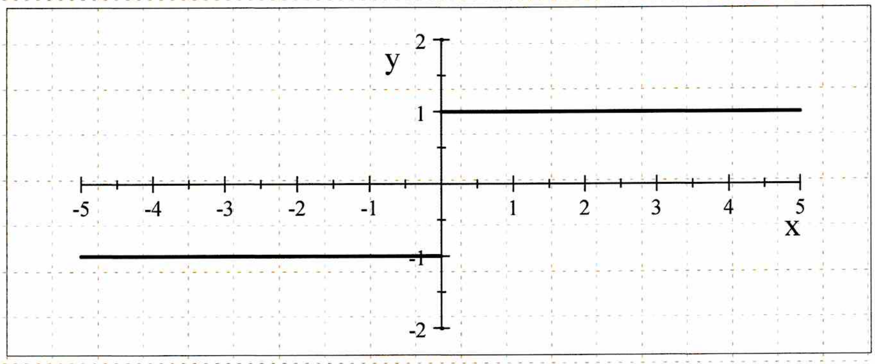
  where $g(x) \neq 0$.
- $|f|(x) := |f(x)| \quad \forall x \in X$

$X = (a, b) \subset \mathbb{R}$

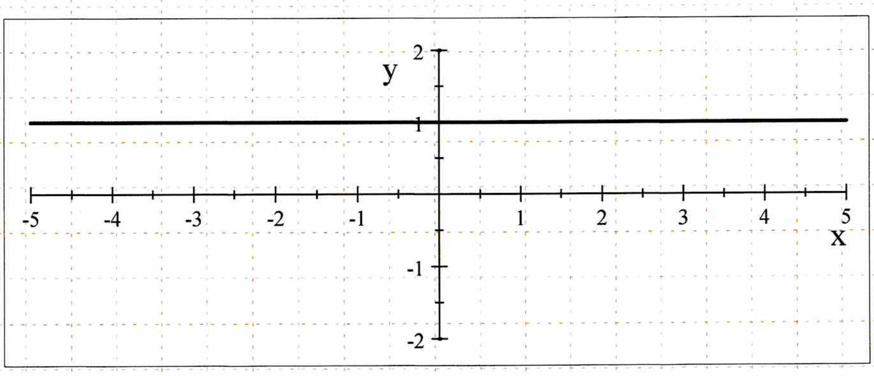


ex.

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$$



$|f|(x) = 1$ for all $x \in \mathbb{R}$



Def

$\{a_n\}, \{b_n\} \subset \mathbb{R}$

$\alpha \in \mathbb{R}$

- $\{a_n\} + \{b_n\} = \{a_n + b_n\}$ (和)
- $\alpha\{a_n\} = \{\alpha a_n\}$ (定数倍)

$$\{a_n + b_n\} = \{a_1 + b_1, a_2 + b_2, a_3 + b_3, \cdots\}$$

$$\{\alpha a_n\} = \{\alpha a_1, \alpha a_2, \alpha a_3, \cdots\}$$

$$\{a_n - b_n\} = \{a_1 - b_1, a_2 - b_2, a_3 - b_3, \cdots\}$$

※ $\{a_n b_n\}$ なども同様に定義される。

ex

$\{a_n\} = \{1, -1, 1, -1, \cdots\}$

$\{b_n\} = \{-1, 1, -1, 1, \cdots\}$

Then,

- $\{a_n\} + \{b_n\}$

$$= \{a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3,\ \cdots\}$$
$$= \{1 + (-1),\ (-1) + 1,\ 1 + (-1),\ \cdots\}$$
$$= \{0, 0, 0, \cdots\}$$

- $-\{a_n\}$

$$= \{-a_n\}$$
$$= \{-1, 1, -1, 1, \cdots\}$$
$$= \{b_n\}$$

- $\{a_n\} - \{b_n\}$

$$= \{2, -2, 2, -2, \cdots\}$$

Real Numbers and Sequences

1. 絶対値の基本性質(A1)–(A3)をもとに，以下を証明せよ．その際，(A1)–(A3)のどれをどこで用いたか明示せよ．

(1) $|x_1 + x_2 + x_3 + x_4| \leq |x_1| + |x_2| + |x_3| + |x_4|$

(2) $||x| - |y|| \leq |x - y|$

2. 数列

$$\{a_n\} = \{2, -1, 2, -1, \cdots\},$$
$$\{b_n\} = \{-1, 1, -1, 1, \cdots\}$$

について，以下の数列を求めよ．

(1) $\{a_n\} - \{b_n\}$ (2) $-2\{a_n\} + \{b_n\}$ (3) $\{a_n b_n\}$ (4) $\{a_n\} + \{|b_n|\}$