Open and closed sets (1)

Def

(X, d) MS

$A \subset X$ open set in X

$\Longleftrightarrow \ {}^{\forall} x \in A, \ {}^{\exists} r > 0 : S_r(x) \subset A$

$\neg$ (否定)

$A \ (\subset X)$: not open in X

$\Longleftrightarrow \ {}^{\exists} x \in A : {}^{\forall} r > 0, \ S_r(x) \not\subset A$

- X, ϕ : open in X.

ex

$X = \mathbb{R}$

$a, b \in \mathbb{R} : a < b$

$\Rightarrow (a, b)$: open in $\mathbb{R}$.

Proof

Let $x \in (a, b)$.

i.e. $a < x < b$.

Define $r \equiv \min\{b - x, x - a\} > 0$.

Then, $S_r(x) = (x - r, x + r) \subset (a, b)$.

$\therefore \forall x \in (a, b), \exists r > 0 : S_r(x) \subset (a, b)$.

$\therefore (a, b)$ is open in $\mathbb{R}$.

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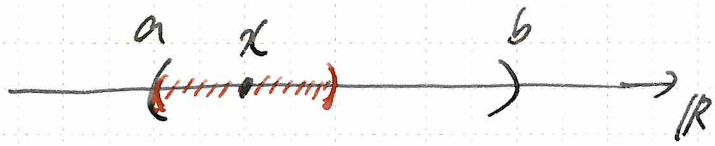


$S_r(x)$

where $r = x - a > 0$.

- (a, ∞) open in $\mathbb{R}$
- $(-\infty, a) \cup (b, \infty)$ 〃
- $\mathbb{R} \setminus \{a\}$ 〃

Def

$\boxed{\begin{array}{l} (X, d)\ \text{MS} \\ A \subset X\ \underline{\text{closed}}\ \text{in } X. \\ \Leftrightarrow A^c : \text{open in } X. \end{array}}$

ex

- X, ϕ : closed in X.

  ($\because$) $X = \phi^c$, $\phi = X^c$.

- $X = \mathbb{R}$
  - $[0, 1]$: closed in $\mathbb{R}$.
  - $[0, \infty)$: 〃
  - $(-\infty, 0] \cup [1, 2]$: 〃
  - $\{0\}$: closed in $\mathbb{R}$.
  - $\mathbb{N}$ $(\subset \mathbb{R})$: 〃

※ A $(\subset X)$: open

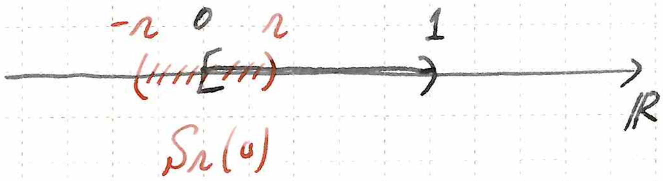
$\Leftrightarrow A^c$ $(\subset X)$: closed

- $X = \mathbb{R}$

  $[0,1)$ is not open in $\mathbb{R}$.

  i.e. $\exists x \in [0,1) : \forall r > 0,\ S_r(x) \not\subset [0,1)$

  $(\because)\ \exists 0 \in [0,1) : \forall r > 0,\ S_r(0) \not\subset [0,1)$.

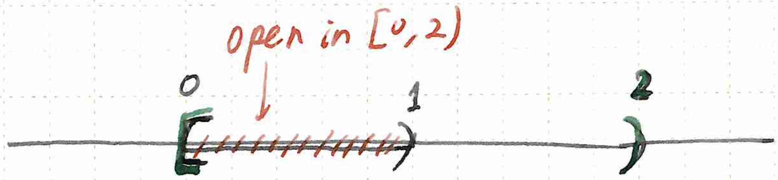


- $X = [0,2)$

  $[0,1)$ is open in X.

  i.e. $\forall x \in [0,1),\ \exists r > 0 : S_r(x) \subset [0,1)$,

  where $S_r(x) = \{ y \in [0,2) \mid |x-y| < r \}$.

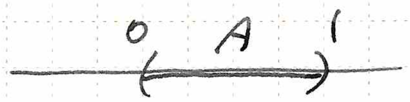


※ open (closed) set の定義において、全空間 X に関する情報は open sphere $S_r(x)$ に入っている。

ex

$X = \mathbb{R}$

$A = (0,1)$ open in $\mathbb{R}$.



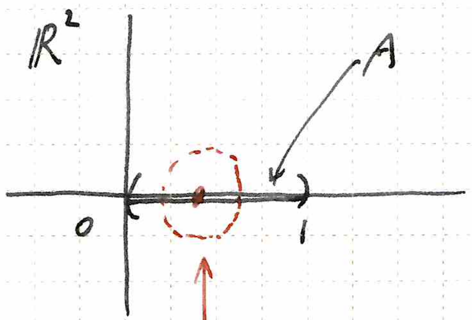
ex

$X = \mathbb{R}^2$

$A = \{(x,y) \in \mathbb{R}^2 \mid 0 < x < 1,\ y = 0\}$

Then, A is not open in $\mathbb{R}^2$.

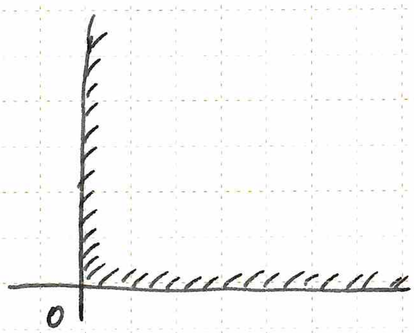
i.e. $\exists (x,y) \in A : \forall r > 0,\ S_r((x,y)) \not\subset A$



$S_r((x,y)) \not\subset A$

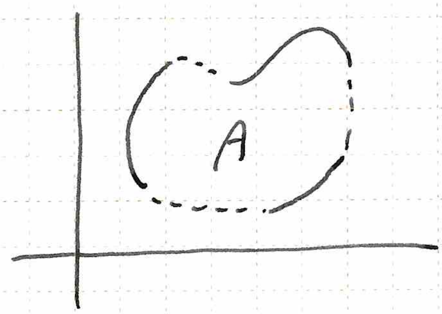
ex

$X = \mathbb{R}^2$



$A = \{(x,y) \in \mathbb{R}^2 \mid x, y \geq 0\}$: closed in $\mathbb{R}^2$.

ex



- A is $\begin{cases} \text{not open} \\ \text{not closed} \end{cases}$ in $\mathbb{R}^2$
- A is open and closed in A.

(X, d) MS

$x \in X$, $r > 0$

$\Rightarrow S_r(x) = \{y \in X \mid d(x, y) < r\}$

open in X.

<u>Proof</u>

Our aim is to prove that

$\underline{\forall z \in S_r(x),\ \exists \varepsilon > 0 : S_\varepsilon(z) \subset S_r(x).}$

Let $z \in S_r(x)$.

Then, $d(x, z) < r$. — $(*)$

Define $\varepsilon = r - d(x, z)$. — $(**)$

From $(*)$, $\varepsilon > 0$.

Let $y \in S_\varepsilon(z)$.

i.e. $d(z, y) < \varepsilon$. — $(*3)$

We show that $\underline{y \in S_r(x).}$

i.e $d(x, y) < r$.

It holds that

$$\begin{aligned} d(x,y) &\le d(x,z) + d(z,y) && (*3) \\ &< d(x,z) + \varepsilon && (**) \\ &= r \end{aligned}$$

Thus, we obtain $y \in S_r(x)$.

$\therefore\ ^{\forall} z \in S_r(x)$,

$\exists\, \varepsilon > 0 : S_\varepsilon(z) \subset S_r(x)$.

$\therefore\ S_r(x)$: open in X.

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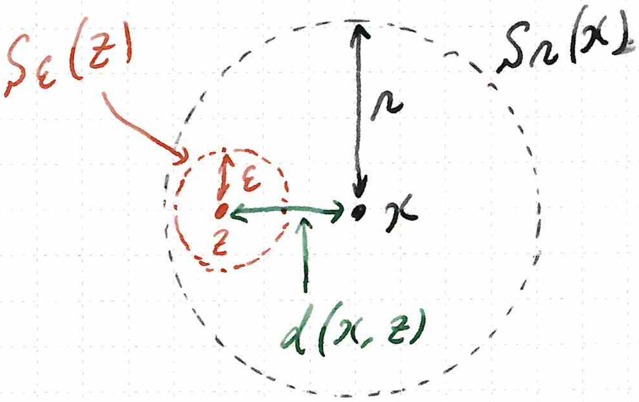

$\varepsilon = r - d(x,z) > 0.$

$X \neq \phi$ with the discrete metric

i.e. $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$

$A \subset X$

$\Longrightarrow A$ is open and closed in X.

<u>Proof.</u>

<u>$A (\subset X)$: open</u>

i.e. $\forall x \in A, \exists r > 0 : S_r(x) \subset A$

Let $x \in A$.

Choose $r \in (0, 1]$.

Then, $S_r(x) = \{x\}$.

We have $S_r(x) = \{x\} \subset A$.

$\therefore \forall x \in A, \exists r > 0 : S_r(x) \subset A$. $\lrcorner$

<u>$A (\subset X)$: closed.</u>

i.e. $A^c (\subset X)$: open

OK.

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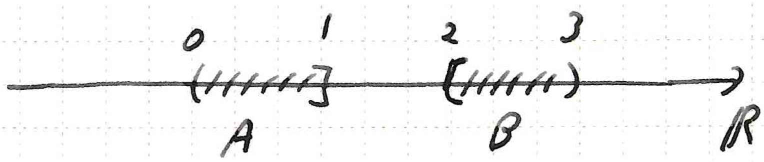
ex

$X = (0,1] \cup [2,3)$

subspace of $\mathbb{R}$

$A = (0,1]$

$B = [2,3)$



Then, A is open in X.

Similarly, B is open in X.

As $A^c = B$ is open in X, A is closed in X.

$\therefore$ A is open and closed in X.

B is 〃 〃

Th

(X, d) MS

$A \subset X$

$\Rightarrow$ Equivalent

① A is open in X.

② $\forall x \in A, \{x_n\} \subset A^c, x_n \not\to x$

Proof

① $\Rightarrow$ ②

Suppose for the sake of contradiction

that $\exists x \in A, \{x_n\} \subset A^c : x_n \to x$.

As $x \in A$, it follows from ① that

$\exists r > 0 : S_r(x) \subset A$.

As $x_n \to x$, for $r > 0$,

$\exists n_0 \in \mathbb{N} : n \geq n_0 \Rightarrow x_n \in S_r(x) \subset A$.

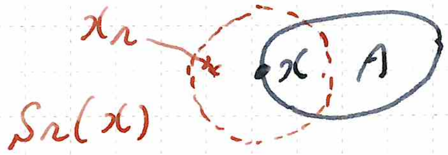
This contradicts $\{x_n\} \subset A^c$. ⌟

② ⇒ ①

Suppose by way of contradiction that A is not open in X.

Then, $\exists x \in A : \forall r > 0,\ S_r(x) \not\subset A$.

$\therefore \exists x \in A : \forall r > 0,\ \exists x_r \in S_r(x) \cap A^c$.



$\therefore \exists x \in A : \forall r > 0,\ \exists x_r \in A^c : d(x, x_r) < r$.

$\therefore \exists x \in A : \forall r = \frac{1}{n} > 0\ (n \in \mathbb{N}),$

$$\exists x_n \in A^c : d(x, x_n) < \frac{1}{n}.$$

$\therefore \exists x \in A,\ \{x_n\} \subset A^c : x_n \to x$.

This contradicts ②.

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※ $r = \frac{1}{n}\ (n \in \mathbb{N})$ として点列の話に帰着させている。

Remark

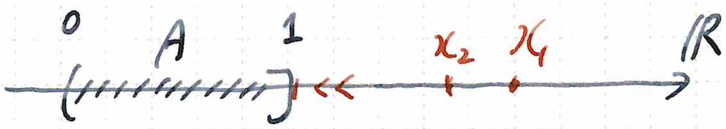
(X, d) MS

$A \subset X$

$\Longrightarrow$ Equivalent

① A is not open

② $\exists x \in A, \{x_n\} \subset A^c : x_n \to x$



not open

$\exists 1 \in A$:

$\exists \{x_n\} \subset A^c : x_n \to 1.$

Th

(X, d) MS

$A \subset X$

$\Rightarrow$ <u>Equivalent</u>

① A is closed in X.

② $\{x_n\} \subset A : x_n \to x \in X$

$\Rightarrow x \in A$

Proof

① $\Leftrightarrow$ A^c is open in X.

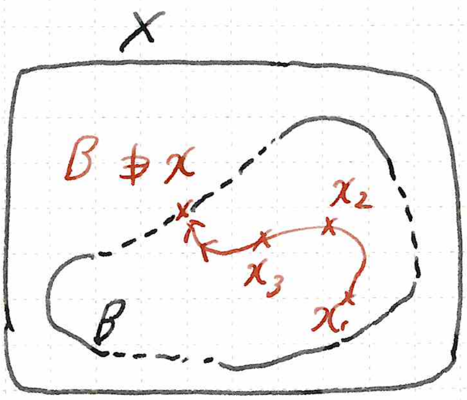
$\Leftrightarrow \forall x \in A^c (\subset X),\ \{x_n\} \subset A^{cc} = A,$

$x_n \not\to x$

$\Leftrightarrow \{x_n\} \subset A,\ x_n \to x \in X$

$\Rightarrow x \in A$

$\Leftrightarrow$ ②. //



B is <u>not</u> closed in X.

$\Leftrightarrow \exists \{x_n\} \subset B :$

$\begin{cases} x_n \to x \in X \\ x \notin B \end{cases}$

exercise

$[a, \infty)$: closed in $\mathbb{R}$.

Proof

Let $\{x_n\} \subset [a, \infty) : x_n \to x \in \mathbb{R}$.

i.e. $\forall n \in \mathbb{N},\ a \leq x_n$

We prove that $x \in [a, \infty)$.

i.e. $a \leq x$

As $x_n \to x$, we obtain

the desired result. //

Review

$\{a_n\}, \{b_n\} \subset \mathbb{R}$

$a_n \to a$

$b_n \to b$

$a_n \leq b_n$

$\Rightarrow a \leq b$

ex

$(0,1]$ is not closed in $\mathbb{R}$.

i.e. $\exists \{x_n\} \subset (0,1]$: $\begin{cases} x_n \to x \in \mathbb{R} \\ x \notin (0,1] \end{cases}$

Indeed, let $x_n = \frac{1}{n}$ $(n \in \mathbb{N})$.

Then, • $\{x_n\} \subset (0,1]$

• $x_n \to 0 \in \mathbb{R}$

However, $0 \notin (0,1]$.

Open and closed sets (1)

1. 距離空間における開集合の定義を述べ，$\mathbb{R}$において(a,∞)が開集合であることを証明せよ.

2. Xを距離空間，Aをその部分集合とする. Aが開集合であることは，A^Cが閉集合であることと同値である. なぜか？

3. 距離空間において，開球が開集合であることを示せ.

4. 離散距離空間の任意の部分集合は開集合かつ閉集合であることを示せ.

5. $X = (0,1] \cup [2,\infty)$を$\mathbb{R}$の部分空間，$A = (0,1]$, $B = [2,\infty)$とする. このとき，AとBはともにXの開集合かつ閉集合となる. これを示せ.

6. Xを距離空間，Aをその部分集合とする. このとき，Aが開集合であることと，どんな$x \in A$と点列$\{x_n\} \subset A^C$についても，$\{x_n\}$はxに収束しないことは同値である. これを示せ.

7. Xを距離空間，Aをその部分集合とする. 問題6より，Aが開集合ではないことと同値な（点列を用いた）条件を答えよ.

8. Xを距離空間，Aをその部分集合とする.

(1) Aが（Xにおいて）閉集合であることは，

$$A\text{の点列}\{x_n\}\text{について，}x_n \to x \in X\text{ならば，}x \in A$$

という条件と同値である(この条件で特徴付けられる). このことを示せ.

(2) 開集合，閉集合というのは，どこを全空間とみなすかによって変わってくるはずである. 上の閉集合の特徴付けの中でどの部分に全空間に関する情報が入っているか？

(3) Aが（Xにおいて）閉集合ではないということは，どう特徴付けられるか？ 論理式とともに図や具体例も併用して説明せよ.

9. 距離空間Xの1点集合$\{x\}$は閉集合である. このことを示せ.