Limits (2)

$\{a_n\} \subset \mathbb{R}$ convergent

i.e. $\exists a \in \mathbb{R} : \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} :$

$$n \geqq n_0 \Rightarrow |a_n - a| < \varepsilon \quad \text{—(*)}$$

$\Longrightarrow \{a_n\}$: bdd

i.e. $\exists M \geqq 0 : \forall n \in \mathbb{N}, |a_n| \leqq M$

Proof.

Define $M = \max\{|a_1 - a|, \cdots, |a_{n_0 - 1} - a|, \varepsilon\} \geqq 0$.

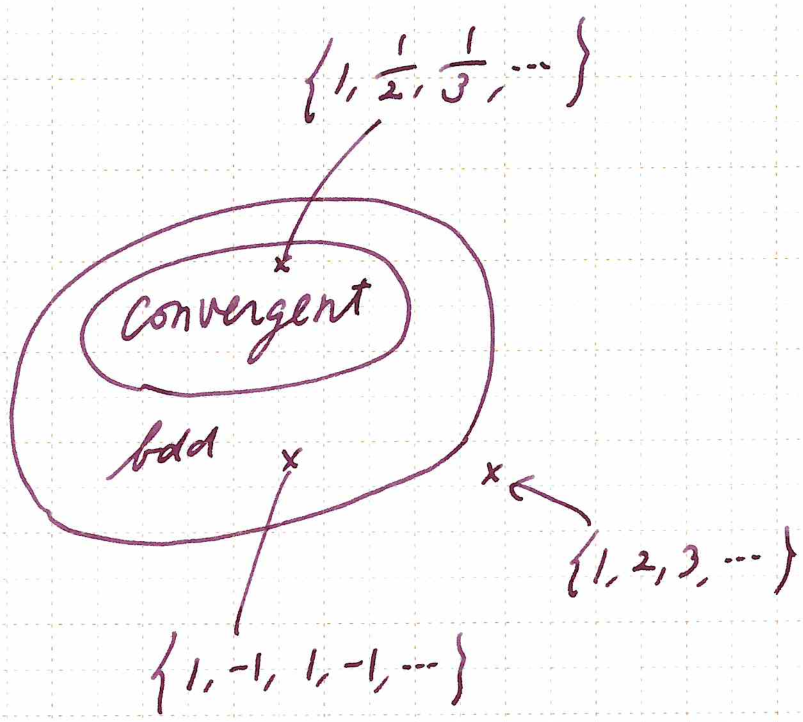
Then, $\forall n \in \mathbb{N}, |a_n - a| \leqq M$ from (*).

Hence, we have

$$\begin{aligned} |a_n| &= |a_n - a + a| \\ &\leqq |a_n - a| + |a| \\ &\leqq M + |a| \equiv M'. \end{aligned}$$

$\therefore \exists M' \geqq 0 : \forall n \in \mathbb{N}, |a_n| \leqq M'$.

//


$\{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$
convergent
bdd
$\{1, 2, 3, \cdots\}$
$\{1, -1, 1, -1, \cdots\}$

Review

(A1) $|x| \geqq 0$; $|x| = 0 \Leftrightarrow x = 0$

(A2) $|\alpha x| = |\alpha|\,|x|$

(A3) $|x+y| \leqq |x| + |y|$

- $\big||a| - |b|\big| \leqq |a-b|$.

- Equivalent

$$\boxed{\begin{array}{l} ① \ a_n \to a \\ ② \ a_n - a \to 0 \\ ③ \ |a_n - a| \to 0 \end{array}}$$

- $$\boxed{\begin{array}{l} \{b_n\}, \{c_n\} \subset \mathbb{R} \\ 0 \leqq b_n \leqq c_n \quad \forall n \in \mathbb{N} \\ c_n \to 0 \\ \Rightarrow b_n \to 0 \end{array}}$$

$$\boxed{\begin{array}{l} \{a_n\} \subset \mathbb{R} \\ a_n \to a \\ \Rightarrow |a_n| \to |a| \end{array}}$$

Proof.

We show that $\underline{||a_n| - |a|| \to 0}$.

As $a_n \to a$ is assumed, $|a_n - a| \to 0$.

Thus, we obtain

$$0 \leq ||a_n| - |a|| \leq |a_n - a| \to 0.$$

$\therefore |a_n| \to |a|$.

//

Remark

$$\boxed{\begin{aligned} & a_n \to a \\ \Rightarrow\ & |a_n| \to |a| \end{aligned}} \longleftrightarrow a = \lim a_n$$

$$|a| = \lim |a_n| \longleftarrow \text{絶対値}\ |\cdot|\ \text{をとってから}\ \lim$$

$$\|$$

$$|\lim a_n| \longleftarrow \lim\ \text{をとってから}\ |\cdot|$$

$$\boxed{\begin{aligned} & \{a_n\} \subset \mathbb{R} \text{ convergent} \\ \Rightarrow\ & \lim |a_n| = |\lim a_n| \end{aligned}}$$

絶対値 $|\cdot|$ と極限 $\lim$ の
順序交換が可能!

ex

逆は言えない！

Consider $\{a_n\} = \{1, -1, 1, -1, \cdots\}$,

which is not convergent.

Then, $\{|a_n|\} = \{1, 1, 1, \cdots\}$.

$\therefore \lim |a_n| = 1$.

On the other hand,

$|\lim a_n|$ does not exist.

$$\boxed{\begin{array}{l} \{a_n\}, \{b_n\} \subset \mathbb{R} \\ a_n \to a \\ b_n \to b \\ \Rightarrow a_n + b_n \to a + b \end{array}}$$

Proof.

Our goal is to prove that

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : n \geq n_0 \Rightarrow |(a_n + b_n) - (a+b)| < \varepsilon.$

Let $\varepsilon > 0$.

As $a_n \to a$, for $\varepsilon/2 > 0$,

$\exists n_1 \in \mathbb{N} : n \geq n_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2}.$

As $b_n \to b$, for $\varepsilon/2 > 0$,

$\exists n_2 \in \mathbb{N} : n \geq n_2 \Rightarrow |b_n - b| < \frac{\varepsilon}{2}.$

Define $n_0 = \max\{n_1, n_2\} \in \mathbb{N}$.

Let $n \geq n_0$.

It holds that

$$|(a_n + b_n) - (a+b)|$$
$$\leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

//

Remark

$a_n \to a \quad (a = \lim a_n)$

$b_n \to b \quad (b = \lim b_n)$

$\Rightarrow a_n + b_n \to a+b$

$\quad (a+b = \lim (a_n + b_n))$

$\Updownarrow$

$\{a_n\}, \{b_n\} \subset \mathbb{R}$ convergent

$\Rightarrow \lim (a_n + b_n) = \lim a_n + \lim b_n$

和と極限の順序交換が可能！

$\{a_n\}, \{b_n\}$: convergent

$\uparrow$

This assumption is indispensable.

<u>ex</u>

$\{a_n\} = \{1, -1, 1, -1, \cdots\}$

$\{b_n\} = \{-1, 1, -1, 1, \cdots\}$

Then, $\{a_n + b_n\} = \{0, 0, 0, \cdots\}$.

Therefore, $\lim(a_n + b_n) = 0$.

On the other hand,

$\nexists \lim a_n, \lim b_n$.

(The RHS $(\lim a_n + \lim b_n)$ can not be defined.)

$$\boxed{\begin{array}{l} \{a_n\} \subset \mathbb{R} \\ a_n \to a \\ \alpha \in \mathbb{R} \\ \Rightarrow \alpha a_n \to \alpha a \end{array}}$$

<u>Proof</u>

(i) If $\alpha = 0$, then

$\{\alpha a_n\} = \{0, 0, 0, \cdots\}$.

In this case, we have $\alpha a_n \to \alpha a = 0$. ⌟

(ii) Assume that $\alpha \neq 0$.

Then, $|\alpha| > 0$.

We show that

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : n \geqq n_0 \Rightarrow |\alpha a_n - \alpha a| < \varepsilon.$$

As $a_n \to a$, for $\varepsilon / |\alpha| > 0$,

$$\exists n_0 \in \mathbb{N} : n \geqq n_0 \Rightarrow |a_n - a| < \frac{\varepsilon}{|\alpha|}. \quad \text{—}(*)$$

Therefore,

$$\begin{aligned} n \geqq n_0 \Rightarrow & |\alpha a_n - \alpha a| \\ & = |\alpha||a_n - a| \overset{(*)}{<} \varepsilon. \end{aligned}$$

//

※ 定数倍と極限の順序交換が可

$\{a_n\}, \{b_n\} \subset \mathbb{R}$ convergent
$\alpha, \beta \in \mathbb{R}$
$\Rightarrow \lim (\alpha a_n + \beta b_n)$
$= \alpha \cdot \lim a_n + \beta \cdot \lim b_n$

Proof

As $\{a_n\}$ and $\{b_n\}$ are convergent, so are $\{\alpha a_n\}$ and $\{\beta b_n\}$.

We have

$$
\begin{aligned}
&\lim (\alpha a_n + \beta b_n) \\
&= \lim (\alpha a_n) + \lim (\beta b_n) \\
&= \alpha \cdot \lim a_n + \beta \cdot \lim b_n .
\end{aligned}
$$

//

※ lim の線型性

$\{a_n\}, \{b_n\}, \{c_n\} \subset \mathbb{R}$ convergent

$\alpha, \beta, \gamma \in \mathbb{R}$

$\Rightarrow \lim(\alpha a_n + \beta b_n + \gamma c_n)$

$= \alpha \cdot \lim a_n + \beta \cdot \lim b_n + \gamma \cdot \lim c_n$

$\{a_n\}, \{b_n\} \subset \mathbb{R}$ convergent

$\Rightarrow \lim(a_n - b_n) = \lim a_n - \lim b_n$

ex

$\{a_n\} = \{b_n\} = \{1, 0, 1, 0, \cdots\}$

Then, $\{a_n - b_n\} = \{0, 0, 0, \cdots\}$.

$\therefore \lim(a_n - b_n) = 0$

However, $\nexists \lim a_n, \lim b_n$.

$$\boxed{\begin{array}{l} \{a_n\} \subset \mathbb{R} \text{ bdd} \\ \{b_n\} \subset \mathbb{R} : b_n \to 0 \\ \Rightarrow a_n b_n \to 0 \end{array}}$$

<u>Proof</u>

As $\{a_n\}$ is bdd,

$$\exists M \geqq 0 : \forall n \in \mathbb{N},\ |a_n| \leqq M.$$

Thus, we obtain

$$\begin{aligned} 0 \leqq |a_n b_n| & \\ = |a_n||b_n| & \\ \leqq M|b_n| & \longrightarrow 0. \end{aligned}$$

$\therefore a_n b_n \to 0.$

//

- the hypothesis $\{a_n\}$ is bdd is indispensable.

<u>ex</u> $\{a_n\} = \{1, 2, 3, \cdots, n, \cdots\}$

$\{b_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\}$

Then, $b_n \to 0$

However, $a_n b_n = 1 \ (\forall n \in \mathbb{N})$.

$\therefore a_n b_n \not\to 0$

<u>ex</u> $\{a_n\} = \{n^2\}$

$\{b_n\} = \{\frac{1}{n}\}$

Then, $b_n \to 0$.

However, $\left(\begin{array}{l} a_n b_n \to \infty \\ a_n b_n \not\to 0. \end{array} \right.$

$\{a_n\}, \{b_n\} \subset \mathbb{R}$ convergent
$\Rightarrow \lim a_n b_n = \lim a_n \cdot \lim b_n$

Proof

Let $\begin{cases} a = \lim a_n \in \mathbb{R} \\ b = \lim b_n \in \mathbb{R}. \end{cases}$

We demonstrate that $a_n b_n \to ab$.

As $\{a_n\}$ is convergent, it is bdd.

Thus, we have

$$
\begin{aligned}
&|a_n b_n - ab| \\
&= |a_n b_n - a_n b + a_n b - ab| \\
&\leq |a_n b_n - a_n b| + |a_n b - ab| \\
&= \underbrace{|a_n|}_{\text{bdd}} \underbrace{|b_n - b|}_{\to 0} + |b| \underbrace{|a_n - a|}_{\to 0} \\
&\to 0.
\end{aligned}
$$

This means that $a_n b_n \to ab$.

//

We have shown that

$$\boxed{\begin{aligned} &a_n \to a \\ &b_n \to b \\ &\Rightarrow a_n b_n \to ab \end{aligned}}$$

The opposite is not always true.

ex

Let $\begin{cases} \{a_n\} = \{b_n\} = \{1, -1, 1, -1, \cdots\} \\ a = 2 \\ b = 1/2. \end{cases}$

Then, $a_n b_n = 1 \longrightarrow 1 = ab$.

However, $\{a_n\}$ and $\{b_n\}$ are not convergent.

$\{a_n\}, \{b_n\} \subset \mathbb{R}$: convergent

$b_n \neq 0 \quad \forall n \in \mathbb{N}$

$\lim b_n \neq 0$

$$\Rightarrow \lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}$$

Proof

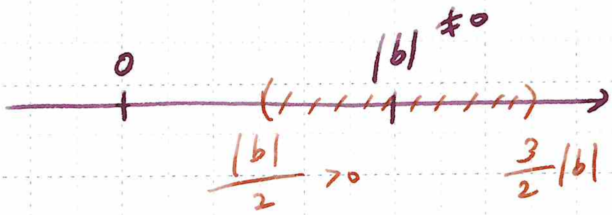
Let $\begin{cases} a \equiv \lim a_n \in \mathbb{R}, \\ b \equiv \lim b_n \in \mathbb{R} \setminus \{0\}. \end{cases}$

Our aim is to prove that $\underline{\dfrac{a_n}{b_n} \to \dfrac{a}{b}}$.

As $b_n \to b \neq 0$,

$$|b_n| \to |b| \neq 0.$$

For $|b|/2 > 0$, $\exists n_0 \in \mathbb{N} : n \geqq n_0 \Rightarrow \dfrac{|b|}{2} < |b_n| \left(< \dfrac{3}{2}|b| \right)$.



Thus, $\dfrac{1}{|b_n|} < \dfrac{2}{|b|} \quad (\forall n \geqq n_0)$.

Let $n \geqq n_0$.

It follows that

$$\left| \frac{a_n}{b_n} - \frac{a}{b} \right|$$

$$= \left| \frac{a_n b - a b_n}{b_n b} \right|$$

$$= \frac{1}{|b_n b|} |a_n b - a b_n|$$

$$\leqq \frac{1}{|b_n||b|} \left(|a_n b - ab| + |ab - a b_n| \right)$$

$n \geqq n_0$

$$\leqq \frac{2}{|b|} \cdot \frac{1}{|b|} \left(|b||a_n - a| + |a||b_n - b| \right)$$

$$\longrightarrow 0.$$

$$\therefore \frac{a_n}{b_n} \to \frac{a}{b}.$$

//

Def.

$\{a_n\} (\subset \mathbb{R})$ diverges to ∞.

$\Longleftrightarrow {}^\forall M>0,\ {}^\exists n_0 \in \mathbb{N};\ n \geqq n_0 \to a_n > M$

$\Longleftrightarrow a_n \to \infty \ (n \to \infty)$

※ $\{a_n\} (\subset \mathbb{R})$ bdd above

$\Longleftrightarrow {}^\exists M>0 :\ {}^\forall n \in \mathbb{N},\ a_n \leqq M$

negation

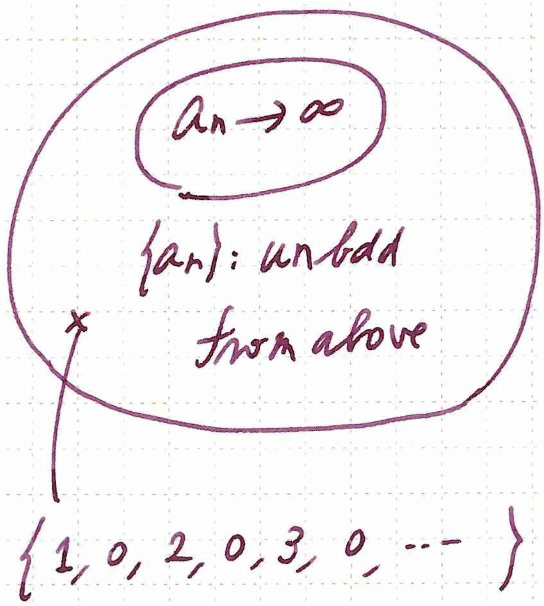
$\{a_n\} (\subset \mathbb{R})$: unbdd above

$\Longleftrightarrow {}^\forall M>0,\ {}^\exists n \in \mathbb{N} :\ a_n > M.$

$\boxed{\begin{array}{l} \{a_n\} \subset \mathbb{R} \\ a_n \to \infty \\ \Rightarrow \{a_n\} \text{ is not bdd above.} \end{array}}$

<u>Proof</u>

It is obvious. //



$\{a_n\}, \{b_n\} \subset \mathbb{R}$

$a_n \leqq b_n \quad \forall n \in \mathbb{N}$ — (*)

$a_n \to \infty$

$\Rightarrow b_n \to \infty$

追い出しの原理

<u>Proof</u>

Let $M > 0$.

We prove that

$\exists n_0 \in \mathbb{N} : n \geqq n_0 \Rightarrow M < b_n$.

As $a_n \to \infty$, for $M > 0$,

$\exists n_0 \in \mathbb{N} : n \geqq n_0 \Rightarrow M < a_n$.

Let $n \geqq n_0$.

From (*), we have $M < a_n \overset{(*)}{\leqq} b_n$.

$\therefore \forall M > 0, \exists n_0 \in \mathbb{N} : n \geqq n_0 \Rightarrow M < b_n$.

//

Limits (2)

1. 収束数列は有界であることを示せ．また，有界だが収束はしない数列の例を挙げよ．

2. 絶対値の性質$||x| - |y|| \leq |x - y|$を復習し，それを用いて数列$\{a_n\}$について，

$$a_n \to a \Rightarrow |a_n| \to |a|$$

を証明せよ．また，逆が言えないことを示す例を挙げよ．

3. 数列$\{a_n\}, \{b_n\}$について，$a_n \to a,\ b_n \to b$とすると$a_n + b_n \to a + b$である．これを示せ．また，仮定$a_n \to a,\ b_n \to b$が必要となることを例を挙げて説明せよ．

4. 数列$\{a_n\}$と実数αについて，$a_n \to a$ならば，$\alpha a_n \to \alpha a$である．これを示せ．また，逆が言えないことを示す例を挙げよ．

5. $a_n \to a,\ b_n \to b,\ \alpha, \beta \in \mathbb{R}$とする．問題3，4の結果を用いて，$\alpha a_n + \beta b_n \to \alpha a + \beta b$を示せ．逆にこの結果を認めると問題3，4の結果が導かれることを示せ．

6. 問題5の結果を用いて，以下を証明せよ．

(1) $\alpha, \beta, \gamma \in \mathbb{R}$とする．このとき，

$$a_n \to a,\ b_n \to b,\ c_n \to c \Rightarrow \alpha a_n + \beta b_n + \gamma c_n \to \alpha a + \beta b + \gamma c$$

となる．

(2) $a_n \to a,\ b_n \to b$とする．このとき，$a_n - b_n \to a - b$がいえる．

7. $\{a_n\}$を有界数列，$\{b_n\}$を0に収束する数列とする．このとき，$a_n b_n \to 0$である．

(1) このことを証明せよ．

(2) $\{a_n\}$が有界でないとき，結論$a_n b_n \to 0$が言えない場合があることを示す例を挙げよ．

(3) $\{a_n\}$は有界数列ではないが，$a_n b_n \to 0$となることもある．そのような数列$\{a_n\}, \{b_n\}$の例を挙げよ．

8. $a_n \to a,\ b_n \to b$とする．このとき，$a_n b_n \to ab$を示せ．また，逆が言えないことを示す例を挙げよ．

9. $a_n \to a,\ b_n \to b$とする．このとき，

$$\frac{a_n}{b_n} \to \frac{a}{b}$$

を示せ．ただし，$b_n, b \neq 0$とする．

10. 数列が上に有界ではないことと$+\infty$に発散することの違いを例を挙げて説明せよ．

11. （追い出しの原理）数列$\{a_n\}, \{b_n\}$が，$a_n \leq b_n\ (n \in \mathbb{N})$かつ$a_n \to \infty$を満たすとする．このとき，$b_n \to \infty$となることを示せ．