Convergence of sequences and Hausdorff spaces

Def.

$(X, \mathfrak{G})$ top. space

$\{x_n\} \subset X$

$x \in X$

- $\{x_n\}$ <u>converges to $x \in X$</u>.

$$\Longleftrightarrow \forall U \in \mathbb{V}(x),\ \exists n_0 \in \mathbb{N} : n \geq n_0 \Longrightarrow x_n \in U$$

- $\{x_n\}$ <u>converges</u>.

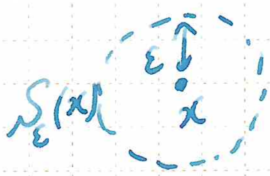
$$\Longleftrightarrow \exists x \in X : \{x_n\} \text{ converges to } x.$$

(X, d) MS

$x_n \to x$

$$\Longleftrightarrow \forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N} : n \geq n_0 \Rightarrow \underline{d(x_n, x) < \varepsilon}$$

i.e. $x_n \in S_\varepsilon(x)$



ex

$X = \{a, b\}$

$\mathcal{G} = \{\phi, \{a\}, X\}$

Then, • $(X, \mathcal{G})$ is a top. space.

• $\mathcal{U}(a) = \{\{a\}, X\}$ $(\subset 2^X)$

• $\mathcal{U}(b) = \{X\}$ $(\subset 2^X)$

(1) Let $\{x_n\} = \{a, a, a, \cdots\}$.

Then, $x_n \longrightarrow a$

$x_n \longrightarrow b$

(2) Let $\{y_n\} = \{b, b, b, \cdots\}$.

Then, $y_n \not\longrightarrow a$

$y_n \longrightarrow b$

(3) Let $\{z_n\} = \{a, b, a, b, \cdots\}$.

Then, $z_n \not\longrightarrow a$

$z_n \longrightarrow b$.

※ In top. space,

$x_n \longrightarrow x$

$x_n \longrightarrow y$

$\not\Longrightarrow x = y$

極限の一意性は言えない！

ex

$X = \mathbb{R}$

$\mathfrak{G} = \{\emptyset, \{0\}, [-1,1], [-2,2], \cdots, [-n,n], \cdots, \mathbb{R}\}$

Then, $(X, \mathfrak{G})$ is a top. space.

Let $\{x_n\} = \{\frac{1}{n}\} (\subset \mathbb{R})$.

- $x_n \not\to 0$.

  i.e. $\exists U \in \mathbb{U}(0) : \forall n \in \mathbb{N},$
  $\exists n' \geqq n : x_{n'} \notin U$

  Letting $U = \{0\} \in \mathbb{U}(0)$, we have that
  $\forall n \in \mathbb{N}, \exists n' \geqq n : x_{n'} \notin \{0\}$.

- $x_n \to 1$

  Note that $\mathbb{U}(1) = \{[-1,1], [-2,2], \cdots, \mathbb{R}\}$.

  Thus, $\forall U \in \mathbb{U}(1), \exists n_0 \in \mathbb{N} :$
  $n \geqq n_0 \Rightarrow x_n \in U$.

- $x_n \to 2$

$(X, \mathfrak{G})$ trivial top.

$\{x_n\} \subset X$

$\Longrightarrow {}^{\forall}x \in X,\ x_n \to x$

(∵) Let $x \in X$.

Then, $\mathcal{U}(x) = \{X\}$.

Hence, ${}^{\forall}U \in \mathcal{U}(x)$, ${}^{\exists}n_0 \in \mathbb{N}$:

$n \geq n_0 \Longrightarrow x \in U \; (= X)$. //

$(X, \mathfrak{G})$ discrete top.

$\{x_n\} \subset X$, $x \in X$

$\Longrightarrow$ Equivalent

① $x_n \to x$

② ${}^{\exists}n_0 \in \mathbb{N}$: $n \geq n_0 \Longrightarrow x_n = x$

$(X, \mathfrak{G})$ top. space

$A \subset X$ closed in X

$\{x_n\} \subset A : x_n \to x \in X$

$\Longrightarrow x \in A$

Proof

Suppose by way of contradiction that

$x \notin A$.

Then, $x \in A^c$.

From the hypothesis that A is closed in X,

A^c is open in X.

Consequently, $A^c \in \mathbb{U}(x)$.

As $x_n \to x$,

$\exists n_0 \in \mathbb{N} : n \geq n_0 \Rightarrow x_n \in A^c$.

This contradicts $\{x_n\} \subset A$.

//

<u>ex</u>

$X = \{a, b\}$

$\mathbb{G} = \{\emptyset, \{a\}, X\}$

$A = \{a\}$

$B = \{b\}$ closed in X.

Then, $A \in \mathbb{G}$ and $B \in \mathbb{F}$.

It holds that

- B is closed.
- $\{y_n\} \subset B : y_n \to y \in X$
  $\Rightarrow y \in B$

In contrast,

- A is not closed.
- $\exists \{x_n\} \subset A : x_n \to x \in X$
  $\not\Rightarrow x \in A$

In a general top. space $(X, \mathcal{G})$,

$$\boxed{\begin{array}{l} A \subset X \text{ closed} \\ \text{i.e. } A^c \in \mathcal{G} \end{array}}$$

$$\Downarrow$$

$$\boxed{\begin{array}{l} \{x_n\} \subset A : x_n \to x \in X \\ \implies x \in A \end{array}}$$

※ In MSs, these two statements are equivalent.

cf.

$(X, \mathfrak{G})$ top. space
$A \subset X$ open in X.
$x \in A$
$\{x_n\} \subset A^c$
$\Longrightarrow x_n \not\to x$

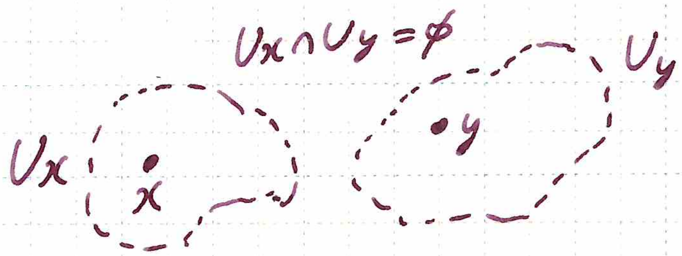
$A \subset X$ open

$\Downarrow$

$x \in A$, $\{x_n\} \subset A^c$
$\Longrightarrow x_n \not\to x$

Def.

$$\boxed{\begin{array}{l} (X, \mathfrak{O}) \text{ top. space} \\ (X, \mathfrak{O}) \ \underline{\text{Hausdorff top. space}} \\ \Leftrightarrow \forall x, y \in X : x \neq y, \\ \qquad \exists U_x \in \mathcal{U}(x),\ U_y \in \mathcal{U}(y) : \\ \qquad\qquad U_x \cap U_y = \phi \end{array}}$$



ex

(X, d) M.S

$\mathfrak{O}$: the set of open sets in (X, d)

$\Rightarrow (X, \mathfrak{O})$: Hausdorff top. space

ex

$X = \mathbb{R}$

$\mathfrak{O} = \{\phi, X\}$ trivial top.

$\Rightarrow (X, \mathfrak{O})$ is not a Hausdorff top. space

$(X, \mathfrak{G})$ Hausdorff top. space

$\{x_n\} \subset X$

$x_n \to x$

$x_n \to y$

$\Rightarrow x = y$

Proof

Suppose to lead a contradiction that

$$x \neq y.$$

As $(X, \mathfrak{G})$ is a Hausdorff top. space,

$$\exists U \in \mathcal{U}(x), V \in \mathcal{U}(y) : U \cap V = \emptyset. \quad \text{— } (*)$$

As $x_n \to x$, for $U \in \mathcal{U}(x)$,

$$\exists n_1 \in \mathbb{N} : n \geqq n_1 \Rightarrow x_n \in U. \quad \text{— } (**)$$

As $x_n \to y$, for $V \in \mathcal{U}(y)$,

$$\exists n_2 \in \mathbb{N} : n \geqq n_2 \Rightarrow x_n \in V. \quad \text{— } (*3)$$

Let $n_0 = \max\{n_1, n_2\} \in \mathbb{N}$.

From $(**)$ and $(*3)$, $x_{n_0} \in U \cap V$.

This contradicts $(*)$.

//

Convergence of sequences and Hausdorff spaces

1. 位相空間$(X, \boldsymbol{G})$内の点列$\{x_n\}$が要素$x \in X$に収束することと収束しないことの定義を述べよ.

2. 密着位相空間$(X, \boldsymbol{G})$ではどんな点列もXの任意の点に収束する. なぜか？

3. $X = \{a, b, c\}$, $\boldsymbol{G} = \{\emptyset, \{a\}, \{a, b\}, X\} (\subset 2^X)$とする. 位相空間$(X, \boldsymbol{G})$内の次の点列が$X$の各点に収束するかしないかを答えよ.
(1) $\{x_n\} = \{a, b, a, b, \cdots\}$ (2) $\{y_n\} = \{b, c, b, c, \cdots\}$

4. $X = \{a, b, c\}$に適当な位相を入れて, aとbには収束するが, cには収束しないような点列の例を造れ.

5. $(X, \boldsymbol{G})$を位相空間, Aをその閉集合とする. このとき,

$$\{x_n\} \subset A, \quad x_n \to x \in X$$
$$\Rightarrow x \in A$$

が成り立つ. このことを証明せよ.

6. ハウスドルフ空間の定義を述べ, 距離空間がハウスドルフ空間であることを示せ.

7. ハウスドルフ空間においては, 点列の極限は（存在するならば）一意に定まる. このことを示せ.

解答

3. (1) $\{x_n\}$はbとcには収束するが, aには収束しない. (2) $\{y_n\}$はcに収束するが, aとbには収束しない.

4. 例えば, $\boldsymbol{G} = \{\emptyset, \{a, b\}, \{c\}, X\}$とし, 点列$\{x_n\} = \{a, b, a, b, \cdots\}$を考えれば, $\{x_n\}$はaとbには収束するが, cには収束しない.