Topological spaces

$X \neq \phi$

$\mathfrak{G} \subset 2^X$

Def

$(X, \mathfrak{G})$ <u>topological space</u>

$\Leftrightarrow$ (G1) $X, \phi \in \mathfrak{G}$

(G2) $G_\mu \in \mathfrak{G} \ (\mu \in M) \Rightarrow \bigcup_{\mu \in M} G_\mu \in \mathfrak{G}$

(G3) $G_1, \cdots, G_n \in \mathfrak{G} \Rightarrow \bigcap_{i=1}^{n} G_i \in \mathfrak{G}$

<u>ex</u>

(X, d) metric space

$\mathfrak{G} = \{G \subset X \mid G \text{ is open in } X.\}$

$\Rightarrow (X, \mathfrak{G})$ top. space

<u>ex</u> (discrete topology)

$X \neq \phi$

$\mathfrak{G} = 2^X$

$\Rightarrow (X, \mathfrak{G})$ top. space

<u>ex</u> (trivial topology)

$X \neq \phi$

$\mathfrak{G} = \{X, \phi\}$

$\Rightarrow (X, \mathfrak{G})$ top. space

ex

$X = \{a, b, c, d, e\}$

$\mathfrak{G}_1 = \{\phi, \{a\}, X\}$

$\mathfrak{G}_2 = \{\phi, \{a\}, \{a, b\}, X\}$

$\mathfrak{G}_3 = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, X\}$

$\mathfrak{G}_4 = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$

$\mathfrak{G}_5 = \{\phi, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$

$\mathfrak{G}_6 = \{\phi, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, X\}$

$\Rightarrow$ $\mathfrak{G}_1, \cdots, \mathfrak{G}_4, \mathfrak{G}_6$: top. spaces

$\mathfrak{G}_5$ is <u>not</u> a top. space.

ex

$X = \mathbb{R}$

$\mathfrak{G} = \{\phi, \{1, 2\}, \{3\}, \{1, 2, 3\}, \mathbb{R}\}$

$\Rightarrow (\mathbb{R}, \mathfrak{G})$: top. space

<u>ex</u>

$\mathbb{R}$

$\mathcal{G}_1 = \{\phi, (-1,1), (-2,2), \cdots, (-n,n), \cdots, \mathbb{R}\}$

$\mathcal{G}_2 = \{\phi, \{0\}, (-1,1), (-2,2), \cdots, (-n,n), \cdots, \mathbb{R}\}$

$\mathcal{G}_3 = \{\phi, [-1,1], [-2,2], \cdots, [-n,n], \cdots, \mathbb{R}\}$

$\Rightarrow (\mathbb{R}, \mathcal{G}_1), (\mathbb{R}, \mathcal{G}_2), (\mathbb{R}, \mathcal{G}_3), (\mathbb{R}, \mathcal{G}_4)$

top. spaces

<u>ex</u>

$\mathbb{R}$

$\mathbb{K} = \{\phi, \{0\}, [0,1), [0,2), \cdots, [0,n), \cdots, \mathbb{R}\}$

$\Rightarrow$ $(\mathbb{R}, \mathbb{K})$ is <u>not</u> a top. space

$(\because)$ $\bigcup_{n=1}^{\infty} [0,n) = [0,\infty) \notin \mathbb{K}$.

//

Def

$\boxed{\begin{array}{l} X \text{ top. space} \\ A \subset X \\ \bullet\ A : \text{open} \\ \quad \Longleftrightarrow A \in \mathfrak{S} \\ \bullet\ A : \text{closed} \\ \quad \Longleftrightarrow A^c \in \mathfrak{S} \end{array}}$

ex (trivial top.)

X top. space
with the trivial top.

Then, $X, \emptyset$: open and closed.

ex (discrete top.)

X top. space
with the discrete top.

Then, ${}^\forall A \in 2^X$, A is open and closed in X.

ex

$X = \{a, b, c, d\}$

$\mathcal{O} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$

Then,

- $\emptyset, \{a\}, \{b\}, \{a, b\}, X$: open in $(X, \mathcal{O})$.
- $X, \{b, c, d\}, \{a, c, d\}, \{c, d\}, \emptyset$

  : closed in $(X, \mathcal{O})$

Th

$(X, \mathfrak{S})$ top. space

$\mathbb{F} = \{F \subset X \mid F \text{ is closed in } X.\}$

$\Rightarrow$ (F1) $X, \phi \in \mathbb{F}$

(F2) $F_\mu \in \mathbb{F}\ (\mu \in M) \Rightarrow \bigcap_{\mu \in M} F_\mu \in \mathbb{F}$

(F3) $F_1, \cdots, F_n \in \mathbb{F} \Rightarrow \bigcup_{i=1}^{n} F_i \in \mathbb{F}$

Proof

(F1) $\underline{X \in \mathbb{F}}$ i.e. $X^c \in \mathfrak{S}$.

As $X^c = \phi \in \mathfrak{S}$, OK.

$\underline{\phi \in \mathbb{F}}$ i.e. $\phi^c \in \mathfrak{S}$.

As $\phi^c = X \in \mathfrak{S}$, OK. ┘

(F2) $\underline{F_\mu \in \mathbb{F}\ (\mu \in M) \Rightarrow \bigcap_{\mu \in M} F_\mu \in \mathbb{F}}$

Let $F_\mu \in \mathbb{F}\ (\mu \in M)$.

i.e. $\forall \mu \in M,\ F_\mu^c \in \mathfrak{S}$.

We show that $\underline{\bigcap_{\mu \in M} F_\mu \in \mathbb{F}}$.

i.e. $\left(\bigcap_{\mu \in M} F_\mu\right)^c \in \mathfrak{S}$

It follows that

$$\left(\bigcap_{\mu \in M} F_\mu\right)^c = \bigcup_{\mu \in M} F_\mu^c \in \mathfrak{S}. \quad ┘$$

(F3) $F_1, \cdots, F_n \in \mathbb{F} \Rightarrow \bigcup_{i=1}^{n} F_i \in \mathbb{F}$

Let $F_i \in \mathbb{F}$ $(i=1,\cdots,n)$.

i.e. $\forall i=1,\cdots,n,\ F_i^c \in \mathfrak{G}$

We show that $\underline{\bigcup_{i=1}^{n} F_i \in \mathbb{F}}$.

i.e $\left(\bigcup_{i=1}^{n} F_i\right)^c \in \mathfrak{G}$

It follows that

$$\left(\bigcup_{i=1}^{n} F_i\right)^c = \bigcap_{i=1}^{n} F_i^c \in \mathfrak{G}.$$

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$(X, \mathfrak{G})$ top. space

(G1) $X, \phi \in \mathfrak{G}$

(G2) $G_\mu \in \mathfrak{G}$ $(\mu \in M) \Rightarrow \bigcup_{\mu \in M} G_\mu \in \mathfrak{G}$

(G3) $G_1, \cdots, G_n \in \mathfrak{G} \Rightarrow \bigcap_{i=1}^{n} G_i \in \mathfrak{G}$

Def.

X top. space

$x \in X$

- $U\ (\subset X)$ <u>neighborhood</u> of x

$\Leftrightarrow \begin{cases} ① \ x \in U \\ ② \ U \in \mathfrak{G} \end{cases}$

- $\mathfrak{U}(x)\ (\subset \mathfrak{G})$ <u>neighborhood system of x</u>

$\Leftrightarrow \mathfrak{U}(x) = \{ U \in 2^X \mid x \in U,\ U \in \mathfrak{G} \}$

ex

$X = \{a, b, c, d, e\}$

$\mathfrak{O} = \{\emptyset, \{a\}, \{a, b\}, X\}$

Then,

$\mathcal{U}(a) = \{\{a\}, \{a, b\}, X\}$ $(\subset 2^X)$

$\mathcal{U}(b) = \{\{a, b\}, X\}$ $(\subset 2^X)$

$\mathcal{U}(c) = \mathcal{U}(d) = \mathcal{U}(e) = \{X\}$ $(\subset 2^X)$

ex

$X = \{a, b, c\}$

$\mathfrak{O}_1 = \{\emptyset, X\}$ trivial top.

$\mathfrak{O}_2 = 2^X$ discrete top.

Then,

- $(X, \mathfrak{O}_1)$

  $\mathcal{U}(a) = \mathcal{U}(b) = \mathcal{U}(c) = \{X\}$ $(\subset 2^X)$

- $(X, \mathfrak{O}_2)$

  $\mathcal{U}(a) = \{\{a\}, \{a, b\}, \{a, c\}, X\}$

ex

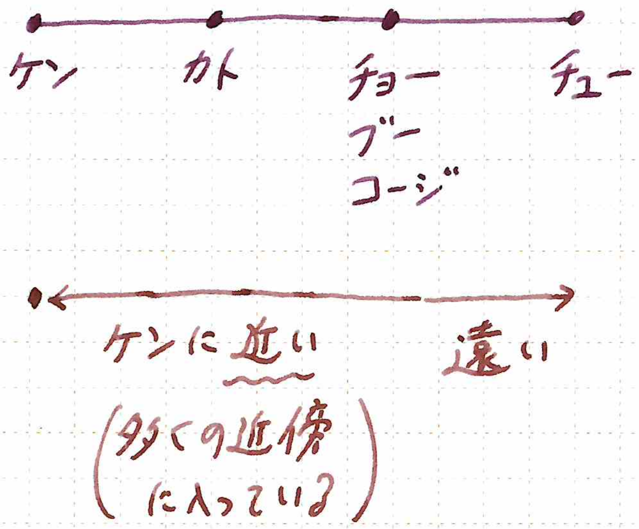
$X = \{$チョー, チュー, ブー, コージ, カト, ケン$\}$

$\mathfrak{O} = \{\phi$, $\{$ケン$\}$, $\{$カト, ケン$\}$,
$\{$チョー, ブー, コージ, カト, ケン$\}$, $X\}$

Then, $(X, \mathfrak{O})$ is a top. space.

・$\mathfrak{U}($ケン$) = \{\{$ケン$\}$, $\{$カト, ケン$\}$,
$\{$チョー, ブー, コージ, カト, ケン$\}$, $X\}$

neighborhood system of ケン.



$(X, \mathfrak{G})$ top. space

$G \subset X$

$\Rightarrow$ Equivalent

① $G \in \mathfrak{G}$ (G is open in X.)

② $\forall x \in G, \exists U_x \in \mathbb{U}(x) : U_x \subset G$

③ $\exists \{U_\lambda\}_{\lambda \in \Lambda} \subset \mathfrak{G} : G = \bigcup_{\lambda \in \Lambda} U_\lambda$

Proof

① $\Rightarrow$ ②

Let $x \in G$.

Define $U_x = G \; (\subset X)$.

As $U_x \; (= G) \in \mathfrak{G}$ and $x \in U_x$,

it follows that $U_x \in \mathbb{U}(x)$.

Furthermore, $U_x \; (= G) \subset G$.

$\therefore \forall x \in G, \exists U_x \in \mathbb{U}(x) : U_x \subset G$. $\lrcorner$

②⇒③

From ②,

$\forall x \in G, \exists U_x \in \mathcal{U}(x) : (x \in) U_x \subset G.$ — (*)

It holds that $G = \bigcup_{x \in G} U_x$.

($\subset$) Let $y \in G$.

From (*), $\exists U_y \in \mathcal{U}(y)$.

Therefore, $y \in U_y \subset \bigcup_{x \in G} U_x$.

($\supset$) Let $y \in \bigcup_{x \in G} U_x$.

i.e. $\exists y \in G : y \in U_x$.

From (*), $y \in U_x \subset G$. $\therefore y \in G$.

③⇒①

From (G2), OK.

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$$\boxed{\begin{array}{l} (G1)\ \phi, X \in \mathfrak{G} \\ (G2)\ G_\mu \in \mathfrak{G}\ (\mu \in M) \Rightarrow \bigcup_{\mu \in M} G_\mu \in \mathfrak{G} \\ (G3)\ G_1, \cdots, G_n \in \mathfrak{G} \Rightarrow \bigcap_{i=1}^{n} G_i \in \mathfrak{G} \end{array}}$$

※ $G = \phi \iff \Lambda = \phi$ in ③.

• (X, d) MS

$d : X \times X \to \mathbb{R}$ (d1)-(d3)

$\downarrow$

$S_r(x) = \{y \in X \mid d(x, y) < r\}$

<u>open sphere</u>

$\downarrow$

<u>**Def.**</u>

$G (\subset X)$: <u>open</u> in X.

$\Leftrightarrow \forall x \in G, \exists r > 0 : S_r(x) \subset G$

$\downarrow$

<u>**Th**</u>

(X, d) MS

$\mathfrak{G} (\subset 2^X)$: the set that collects all open sets in X

$\Rightarrow$ (G1) $\emptyset, X \in \mathfrak{G}$

(G2) $G_\mu \in \mathfrak{G} \ (\mu \in M) \Rightarrow \bigcup_\mu G_\mu \in \mathfrak{G}$

(G3) $G_1, \cdots, G_m \in \mathfrak{G} \Rightarrow \bigcap_{i=1}^{m} G_i \in \mathfrak{G}$

• $(X, \mathfrak{G})$ top. space

$\iff \mathfrak{G}\ (\subset 2^X)$ satisfies $(G1)$-$(G3)$

Def

$G\ (\subset X)$: open in X.

$\iff G \in \mathfrak{G}$

Def

$\mathbb{U}(x) = \{U \in \mathfrak{G} \mid x \in U\}$

neighborhood system of x

Th

$G \subset X$

$\Rightarrow$ Equivalent

① G : open in X

② ${}^\forall x \in G,\ {}^\exists U \in \mathbb{U}(x) : x \in U \subset G$

ex.

$X = \mathbb{R}$

$G = (0,1) \subset \mathbb{R}$

Then,

① $G \ (= (0,1))$: open in $\mathbb{R}$.

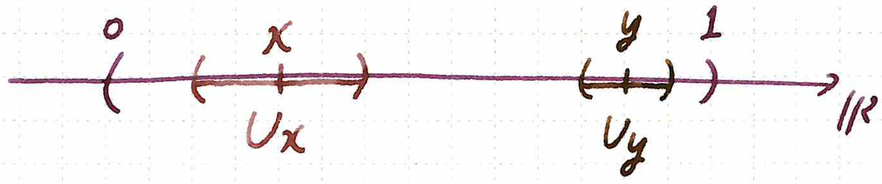
② ${}^{\forall}x \in (0,1)$,

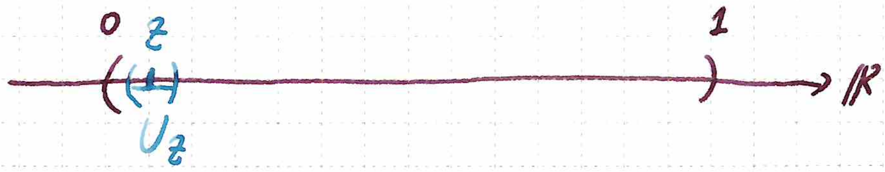
$${}^{\exists}\varepsilon = \min\left\{ \frac{1}{2}|x-0|, \frac{1}{2}|x-1| \right\} > 0;$$

$$U_x \equiv (x-\varepsilon, x+\varepsilon) \subset (0,1).$$

※ This ε depends on x.

③ $(0,1) = \bigcup_{x \in (0,1)} U_x$





ex

$X = \{a, b, c, d, e\}$

$\mathcal{G} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$

Then, $(X, \mathcal{G})$ is a top. space.

Let $G = \{a, b, c\}$.

① G is open in X. (i.e. $G \in \mathcal{G}$)

② For $a \in G$, define $U_a = \{a\} \in \mathcal{U}(a)$.

Then, $U_a \subset G$.

For $b \in G$, define $U_b = \{b\} \in \mathcal{U}(b)$.

Then, $U_b \subset G$

For $c \in G$, define $U_c = \{a, b, c\} \in \mathcal{U}(c)$.

Then, $U_c \subset G$.

③ $$G = \bigcup_{x \in \{a, b, c\}} U_x$$

$$= U_a \cup U_b \cup U_c = \{a, b, c\}.$$

Topological spaces

1. 位相空間の定義と例を述べよ.

2. 距離空間は位相空間の特殊ケースである. それはどういうことか？

3. 実数の集合$\mathbb{R}$に本文中にはない位相を入れて, 位相空間を自分で造ってみよ.

4. $X = \{a,b,c,d,e\}$, $\boldsymbol{G}_1 = \{\emptyset, \{a\}, \{a,b\}, X\}$, $\boldsymbol{G}_2 = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, X\} (\subset 2^X)$とする. 位相空間$(X, \boldsymbol{G}_1)$, $(X, \boldsymbol{G}_2)$の開集合と閉集合を答えよ.

5. 位相空間$(X, \boldsymbol{G})$の閉集合の全体を$\boldsymbol{F}$と書くと, これは3条件

$$\begin{aligned}
&(F1) \quad X, \emptyset \in \boldsymbol{F},\\
&(F2) \quad F_\mu \in \boldsymbol{F}\ (\mu \in M) \Rightarrow \bigcap_{\mu \in M} F_\mu \in \boldsymbol{F},\\
&(F3) \quad F_i \in \boldsymbol{F}\ (i = 1, \cdots, m) \Rightarrow \bigcup_{i=1}^{m} F_\mu \in \boldsymbol{F}
\end{aligned}$$

を満たす. このことを示せ.

6. 問題4の位相空間$(X, \boldsymbol{G}_2)$について, 各点の近傍系を答えよ.

7. $(X, \boldsymbol{G})$を位相空間, Gをその部分集合, $\boldsymbol{U}(x)$をXの要素xの近傍系とする. このとき, 3条件

$$\begin{aligned}
&(1) \quad G \in \boldsymbol{G},\\
&(2) \quad \forall x \in G,\ \exists U_x \in \boldsymbol{U}(x),\ U_x \subset G,\\
&(3) \quad \exists \{U_\lambda\}_{\lambda \in \Lambda} \in \boldsymbol{G},\ G = \bigcup_{\lambda \in \Lambda} U_\lambda
\end{aligned}$$

は同値である. このことを示せ. また, 条件(2)には明示的には位相$\boldsymbol{G}$が現れていないが, $\boldsymbol{G}$に関する情報はこの条件のどこに反映されているか？

8. 問題7により, 位相空間$(X, \boldsymbol{G})$の部分集合Gが開集合であることは, 条件

$$\forall x \in G,\ \exists U_x \in \boldsymbol{U}(x),\ U_x \subset G$$

で特徴付けられることが証明された. 距離空間の場合との理論構成の仕方の違いを確認せよ.

9. 問題4の位相空間$(X, \boldsymbol{G}_2)$について, 各開集合ごとに問題7の(2)が成り立っていることを確認せよ.

10. $X = \{a,b,c,d,e\}$, $\boldsymbol{G} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}, \{a,b,c\}, X\} (\subset 2^X)$とする. 位相空間$(X, \boldsymbol{G})$の開集合$\{a,b,c\}$について, 問題7の(3)に対応する開集合族$\{U_\lambda\}$としてどのようなものをとることができるか？