Basic concepts

$(X, \mathfrak{S})$ top. space

$A \subset X$

- $x \in X$ an <u>interior point</u> of A.

$\Leftrightarrow {}^{\exists} U \in \mathbb{U}(x) : U \subset A$

$\Leftrightarrow {}^{\exists} U \in \mathbb{U}(x) : U \cap A^c = \phi$

$\operatorname{Int} A$: the set of interior points of A

$x \notin \operatorname{Int} A$

$\Leftrightarrow {}^{\forall} U \in \mathbb{U}(x),\ U \cap A^c \neq \phi.$

$$\boxed{\operatorname{Int} A \subset A}$$

<u>Proof</u>

Let $x \in \operatorname{Int} A$.

Then, ${}^{\exists} U \in \mathbb{U}(x) : U \subset A.$ —(*)

i.e. ① $x \in U$
② $U \in \mathfrak{S}$

We show that <u>$x \in A$</u>.

From (*), $x \in U \subset A$.

$\therefore x \in A$.

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- $x \in X$ an <u>exterior point</u> of A

$\iff \exists U \in \mathcal{U}(x) : U \subset A^c$

$\iff \exists U \in \mathcal{U}(x) : U \cap A = \emptyset$

A^e : the set of exterior points of A

$x \notin A^e$

$\iff \forall U \in \mathcal{U}(x),\ U \cap A \neq \emptyset$

$\boxed{A^e \subset A^c}$

- $x \in X$: a <u>boundary point</u> of A

$\iff \forall U \in \mathcal{U}(x),\ U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$

A^b

$x \notin A^b$

$\iff \exists U \in \mathcal{U}(x) : \left(\begin{array}{l} U \cap A = \emptyset \text{ or} \\ U \cap A^c = \emptyset \end{array} \right.$

$\boxed{\begin{array}{c} X = \operatorname{Int} A \cup A^e \cup A^b \\ \text{disjoint} \end{array}}$

ex

$X = \{a, b\}$

$\mathfrak{G} = \{\phi, \{a\}, X\}$

Then, $(X, \mathfrak{G})$: top. space

$\mathcal{U}(a) = \{\{a\}, X\}$

$\mathcal{U}(b) = \{X\}$.

Let $A = \{a\}$.

Then, $A^c = \{b\}$.

(1) $\underline{\operatorname{Int} A = \{x \in X \mid \exists U \in \mathcal{U}(x) : U \subset A\} = ?}$

As $\operatorname{Int} A \subset A$, $b \notin \operatorname{Int} A$.

As $\exists \{a\} \in \mathcal{U}(a) : \{a\} \subset A = \{a\}$.

Thus, $a \in \operatorname{Int} A$.

$\therefore \underline{\operatorname{Int} A = \{a\}}\ (\subset X)$. ┘

(2) $\underline{A^e = \{x \in X \mid \exists U \in \mathcal{U}(x) : U \subset A^c\} = ?}$

As $A^e \subset A^c$, $a \notin A^e$.

Note that $\underline{b \notin A^e}$.

$(\because)$ $\forall U \in \mathcal{U}(b)$, $U \not\subset A^c = \{b\}$.

$\therefore A^e = \phi$. ┘

(3) $A^b = \{x \in X \mid {}^{\forall} U \in \mathbb{U}(x),\ U \cap A \neq \phi,\ U \cap A^c \neq \phi\} = ?$

As $\mathrm{Int}A$, A^e, A^b are a partition of X,

$\mathrm{Int}A = \{a\}$, and $A^e = \phi$, it holds that

$A^b = \{b\}$.

check

- $a \notin A^b$

  i.e. ${}^{\exists} U \in \mathbb{U}(a) : \begin{cases} U \cap A = \phi \text{ or} \\ U \cap A^c = \phi \end{cases}$

  Indeed, ${}^{\exists}\{a\} \in \mathbb{U}(a)$; $U \cap A^c$

  $= \{a\} \cap \{b\} = \phi$. ⌟

- $b \in A^b$

  i.e. ${}^{\forall} U \in \mathbb{U}(b)$, $U \cap A \neq \phi$, $U \cap A^c \neq \phi$.

  OK. ⌟

$(X, \mathfrak{G})$ top. space

$A \subset X$

- $x \in X$ <u>contact point</u> of A 触点

$\Longleftrightarrow \forall U \in \mathbb{U}(x),\ U \cap A \neq \phi$

$\bar{A} = \{x \in X \mid x \text{ is a contact point of } A.\}$

$= \{x \in X \mid \forall U \in \mathbb{U}(x),\ U \cap A \neq \phi\}$

the <u>closure</u> of A 閉包

$x \notin \bar{A}$

$\Longleftrightarrow \exists U \in \mathbb{U}(x) : U \cap A = \phi$

- $x \in X$ <u>accumulation point</u> of A 集積点

$\Longleftrightarrow \forall U \in \mathbb{U}(x),\ U \cap (A \setminus \{x\}) \neq \phi$

$\Longleftrightarrow \forall U \in \mathbb{U}(x),\ U \cap A \cap \{x\}^c \neq \phi$

A^d the <u>derived set</u> of A 導集合

$x \notin A^d$

$\Longleftrightarrow \exists U \in \mathbb{U}(x) : U \cap (A \setminus \{x\}) = \phi$

- $\underset{\sim}{x \in A}$ <u>isolated point</u> of A 孤立点

$$\Leftrightarrow \begin{cases} ① \ x \in A \\ ② \ {}^{\exists}U \in \mathcal{U}(x) : U \cap (A \setminus \{x\}) = \emptyset \end{cases}$$

$$\Leftrightarrow \begin{cases} ① \ x \in A \\ ② \ {}^{\exists}U \in \mathcal{U}(x) : U \cap A \cap \{x\}^c = \emptyset \end{cases}$$

$$\Leftrightarrow \begin{cases} ① \ x \in A \\ ② \ {}^{\exists}U \in \mathcal{U}(x) : U \cap A \subset \{x\} \end{cases}$$

$\mathrm{Iso}\,A$

$x \notin \mathrm{Iso}\,A$

$$\Leftrightarrow \begin{cases} ① \ x \notin A \text{ or} \\ ② \ {}^{\forall}U \in \mathcal{U}(x),\ U \cap (A \setminus \{x\}) \neq \emptyset \end{cases}$$

$$\Leftrightarrow \begin{cases} ① \ x \notin A \text{ or} \\ ② \ {}^{\forall}U \in \mathcal{U}(x),\ U \cap A \cap \{x\}^c \neq \emptyset \end{cases}$$

$$\boxed{\begin{array}{l} \cdot\ A \subset \bar{A} \\ \cdot\ A^d \subset \bar{A} \\ \cdot\ \mathrm{Iso}\,A = A \cap (A^d)^c \end{array}}$$

ex

$X = \{a, b\}$

$\mathcal{G} = \{\phi, \{a\}, X\}$

Then, $(X, \mathcal{G})$ is a top. space.

Note that $\begin{cases} \mathcal{U}(a) = \{\{a\}, X\}; \\ \mathcal{U}(b) = \{X\}. \end{cases}$

Let $A = \{a\}$.

Then, $A^c = \{b\}$.

(1) $\overline{A} = \{x \in X \mid {}^\forall U \in \mathcal{U}(x),\ U \cap A \neq \phi\} = ?$

As $A \subset \overline{A}$, $a \in \overline{A}$.

$b \in \overline{A}$.

i.e. ${}^\forall U \in \mathcal{U}(b)$, $U \cap \{a\} \neq \phi$.

OK.

$\therefore \overline{A} = \{a, b\}$.

(2) $A^d = \{x \in X \mid {}^\forall U \in \mathcal{U}(x),\ U \cap (A \setminus \{x\}) \neq \phi\} = ?$

$a \notin A^d$

i.e ${}^\exists U \in \mathcal{U}(a)$, $U \cap A \cap \{a\}^c = \phi$

i.e. ${}^\exists U \in \mathcal{U}(a)$: $U \cap \{a\} \cap \{a\}^c = \phi$

OK.

$\underline{b \in A^d}$

i.e. $\forall U \in \mathcal{U}(b),\ U \cap (A \setminus \{b\}) \neq \phi$

i.e. $\forall U \in \mathcal{U}(b),\ U \cap (\{a\} \cap \{b\}^c) \neq \phi$

i.e. $\forall U \in \mathcal{U}(b),\ U \cap \{a\} \neq \phi$

OK.

$\therefore \underline{A^d = \{b\}.}$

(3) $\underline{\operatorname{Iso} A = \{x \in A \mid \exists U \in \mathcal{U}(x) : U \cap (A \setminus \{x\}) = \phi\} = ?}$

As $\operatorname{Iso} A \subset A$, $(\operatorname{Iso} A)^c \supset A^c$.

As $b \in A^c$, $b \notin \operatorname{Iso} A$.

$\underline{a \in \operatorname{Iso} A}$

i.e. $\begin{cases} a \in A \\ \exists U \in \mathcal{U}(a) : U \cap (A \setminus \{a\}) = \phi \end{cases}$

OK

$\therefore \underline{\operatorname{Iso} A = \{a\}.}$

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Basic concepts in topological spaces

1. 位相空間における内点, 外点, 境界点, 触点, 集積点, 孤立点の定義を述べよ.

2. 次が成り立つことを納得せよ.
(1) $\mathrm{Int}A \subset A$ (2) $A^e \subset A^C$ (3) $A \subset \overline{A}$ (4) $A^d \subset \overline{A}$

3. $X = \{a, b\}$, $\boldsymbol{G} = \{\emptyset, \{a\}, X\}(\subset 2^X)$とする. 位相空間$(X, \boldsymbol{G})$においてその部分集合$B = \{b\}$の内部 $\mathrm{Int}B$, 外部B^e, 境界B^b, 閉包$\overline{B}$, 導集合B^d, 孤立点の集合 $\mathrm{Iso}B$を求めよ.

4. $X = \{a, b, c\}$, $\boldsymbol{G} = \{\emptyset, \{a\}, \{a, b\}, X\}(\subset 2^X)$とする. 位相空間$(X, \boldsymbol{G})$においてその部分集合$C = \{b, c\}$の内部 $\mathrm{Int}C$, 外部C^e, 境界C^b, 閉包$\overline{C}$, 導集合C^d, 孤立点の集合 $\mathrm{Iso}C$を求めよ.

5. 距離空間におけるこれらの概念と基礎的な結果を復習せよ.

解答

3. $\mathrm{Int}B = \emptyset$, 外部$B^e = \{a\}$, 境界$B^b = \{b\}$, 閉包$\overline{B} = \{b\}$, 導集合$B^d = \emptyset$, 孤立点の集合 $\mathrm{Iso}B = \{b\}$.

4. $\mathrm{Int}C = \emptyset$, 外部$C^e = \{a\}$, 境界$C^b = \{b, c\}$, 閉包$\overline{C} = \{b, c\}$, 導集合$C^d = \{c\}$, 孤立点の集合 $\mathrm{Iso}C = \{b\}$.