Subspaces

$$\boxed{\begin{array}{l}(X,\mathfrak{G})\ \text{top. space}\\ A\subset X\\ \mathfrak{G}^*=\{A\cap G \mid G\in\mathfrak{G}\}\\ \Rightarrow (A,\mathfrak{G}^*)\ \text{top. space}\end{array}}$$

Proof

Note that $G^*\in\mathfrak{G}^* \Leftrightarrow \exists G\in\mathfrak{G} : G^*=A\cap G$.

(G1) $A\in\mathfrak{G}^*$ as $\exists X\in\mathfrak{G} : A=A\cap X$, OK.

$\emptyset\in\mathfrak{G}^*$ as $\exists \emptyset\in\mathfrak{G} : \emptyset=A\cap\emptyset$, OK.

(G2) Let $G^*_\mu\in\mathfrak{G}^*$ $(\mu\in M)$.

i.e. $\forall\mu\in M, \exists G_\mu\in\mathfrak{G} : G^*_\mu=A\cap G_\mu$.

We show that $\bigcup_\mu G^*_\mu\in\mathfrak{G}^*$.

ie $\exists G\in\mathfrak{G} : \bigcup_\mu G^*_\mu = A\cap G$.

Define $G=\bigcup_\mu G_\mu$.

As $G_\mu\in\mathfrak{G}$, we have $G=\bigcup_\mu G_\mu\in\mathfrak{G}$.

Furthermore,

$$\bigcup_\mu G^*_\mu=\bigcup_\mu (A\cap G_\mu)$$

$$=A\cap\left(\bigcup_\mu G_\mu\right)=A\cap G. \quad \lrcorner$$

(G3) Let $G_i^* \in \mathfrak{G}^*$ $(i=1,\cdots,n)$.

i.e. $\forall i=1,\cdots,n,\ \exists G_i \in \mathfrak{G} : G_i^* = A \cap G_i$

We prove that $\underline{\bigcap_{i=1}^{n} G_i^* \in \mathfrak{G}^*}$.

i.e. $\exists G' \in \mathfrak{G} : \bigcap_{i=1}^{n} G_i^* = A \cap G'$.

Define $G' \equiv \bigcap_{i=1}^{n} G_i$.

As $G_i \in \mathfrak{G}$ $(i=1,\cdots,n)$, we have

$$G' \equiv \bigcap_{i=1}^{n} G_i \in \mathfrak{G}.$$

Furthermore, it follows that

$$\begin{aligned}\bigcap_{i=1}^{n} G_i^* &= \bigcap_{i=1}^{n} (A \cap G_i) \\ &= A \cap \left(\bigcap_{i=1}^{n} G_i \right) \\ &= A \cap G'.\end{aligned}$$

//

Def

$(A, \mathfrak{G}^*)$ <u>subspace</u> of $(X, \mathfrak{G})$

$\mathfrak{G}^*$: the <u>relative topology</u> corresponding to $(X, \mathfrak{G})$

ex

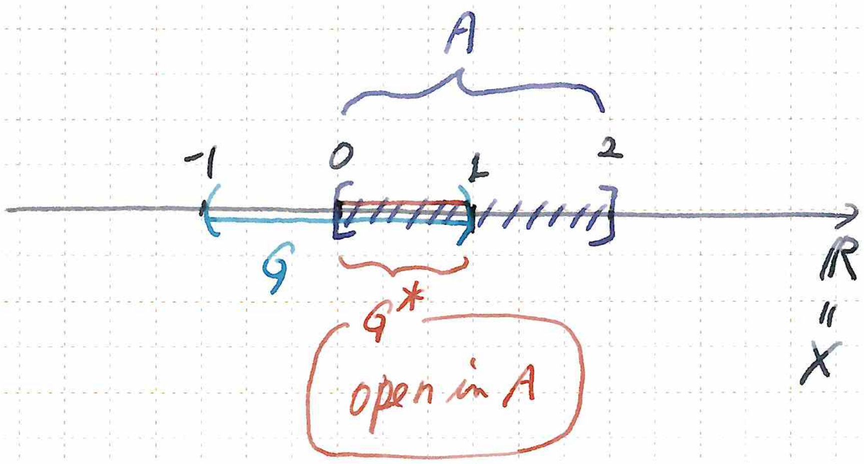
$X = \mathbb{R}$

$A = [0, 2]$

Then, $G^* = [0, 1)$ is open in A.

$(\because)$ $\exists\, G = (-1, 1)$: open in $\mathbb{R}$

$G^* = [0, 1) = G \cap A$. //



ex

$X = \{a, b, c, d, e\}$

$\mathcal{G}_1 = \{\phi, \{a\}, X\}$

$\mathcal{G}_2 = \{\phi, \{a\}, \{a, b\}, X\}$

$\mathcal{G}_3 = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, X\}$

Let $A = \{b, c, d\}$ $(\subset X)$.

- $\mathcal{G}_1^*$: the relative top. corresponding to $(X, \mathcal{G}_1')$

  $\mathcal{G}_1^* = \{\phi, A\}$

- $\mathcal{G}_2^* = \{\phi, \{b\}, A\}$
- $\mathcal{G}_3^* = \{\phi, \{b\}, \{b, c\}, A\}$

Subspaces

1. $(X, \boldsymbol{G})$を位相空間, Aをその空でない部分集合とする. ここで,

$$\boldsymbol{G}^* = \{G^* \subset A : \text{ある}G \in \boldsymbol{G}\text{が存在して}, G^* = G \cap A\text{となる}.\}$$

とおくと, $(A, \boldsymbol{G}^*)$は位相空間になる. このことを示せ.

2. 実数空間$\mathbb{R}$の部分空間$A = [0, 2]$を考える. このとき, $B = [0, 1)$はAにおける開集合である. なぜか?

3. 実数空間$\mathbb{R}$の部分空間$C = (0, 2)$を考える. このとき, $D = (0, 1]$はAにおける閉集合である. なぜか?

4. 集合$X = \{a, b, c, d, e\}$上に位相$\boldsymbol{G} = \{\emptyset, \{e\}, \{e, d\}, \{e, d, c\}, X\}$を入れる. 部分空間$A = \{a, b, c, d\}$の相対位相を答えよ.