Connectedness (1)

Def.

$(X, \mathfrak{G})$ top. space

$A \subset X$

A is not connected.

$\Leftrightarrow {}^{\exists} G_1, G_2 \in \mathfrak{G} \setminus \{\emptyset\}$:

$$\begin{cases} G_1 \cap A,\ G_2 \cap A \neq \emptyset \\ G_1 \cap G_2 = \emptyset \\ A \subset G_1 \cup G_2 \end{cases}$$

$\downarrow$ In particular, $A = X$

Def.

$(X, \mathfrak{G})$ is not connected.

$\Leftrightarrow {}^{\exists} G_1, G_2 \in \mathfrak{G} \setminus \{\emptyset\}$:

$$\begin{cases} G_1 \cap G_2 = \emptyset \\ X = G_1 \cup G_2 \end{cases}$$

ex

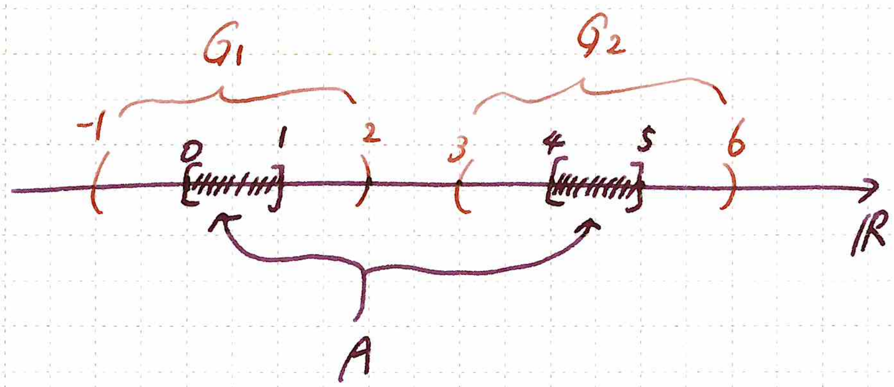
$X = \mathbb{R}$

$A = [0,1] \cup [4,5]$

Then, A is not connected.

(∵) Let $\begin{cases} G_1 = (-1, 2) \\ G_2 = (3, 6) \end{cases}$

Then,
- G_1, G_2 are open in $\mathbb{R}$.
- $G_1, G_2 \neq \phi$
- $G_1 \cap A,\ G_2 \cap A \neq \phi$
- $G_1 \cap G_2 = \phi$
- $A \subset G_1 \cup G_2$.



Review

Def

$(X, \mathfrak{G})$ top. space

$A \subset X$ not connected.

$\Longleftrightarrow \exists G_1, G_2 \subset X, \neq \emptyset$

① $G_1, G_2 \in \mathfrak{G}$

② $G_1 \cap A \neq \emptyset$, $G_2 \cap A \neq \emptyset$

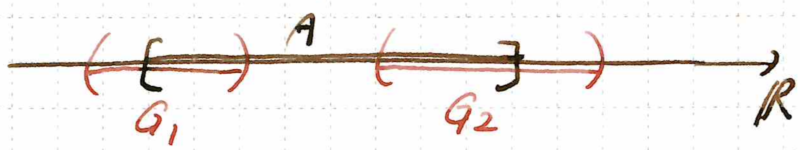
③ $G_1 \cap G_2 = \emptyset$

④ $A \subset G_1 \cup G_2$

④ is indispensable.

$X = \mathbb{R}$

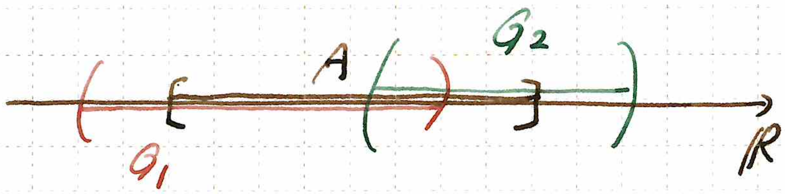
The set A should be "connected".



G_1, G_2 and A satisfy ①-③.

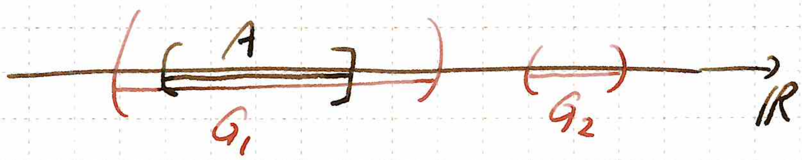
However, they do not satisfy ④.

③ is indispensable.

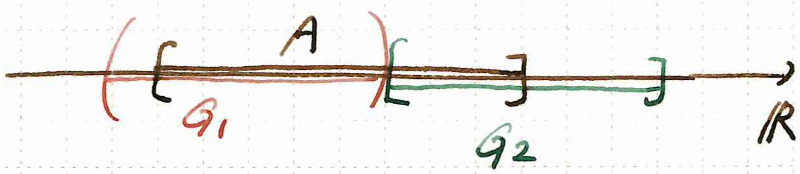


Although G_1, G_2 and A satisfy ①②④, only ③ is bankrupt.

② is indispensable.



① is indispensable.



Def.

$(X, \mathfrak{G})$ top. space

$A \subset X$

A is <u>connected</u>.

$$\Longleftrightarrow \forall G_1, G_2 \in \mathfrak{G} \setminus \{\emptyset\},\ \begin{cases} G_1 \cap G_2 = \emptyset \\ A \subset G_1 \cup G_2 \end{cases} \Rightarrow G_1 \cap A = \emptyset \text{ or } G_2 \cap A = \emptyset$$

Def.

$(X, \mathfrak{G})$ is <u>connected</u>.

$$\Longleftrightarrow \forall G_1, G_2 \in \mathfrak{G} \setminus \{\emptyset\},\ G_1 \cap G_2 \neq \emptyset \text{ or } X \supsetneq G_1 \cup G_2$$

ex

$X = \{a, b, c\}$

$\mathfrak{G}_1 = \{\phi, \{a\}, X\}$

$\mathfrak{G}_2 = \{\phi, \{a\}, \{b, c\}, X\}$

Then, $(X, \mathfrak{G}_1)$, $(X, \mathfrak{G}_2)$ are top. spaces.

- $(X, \mathfrak{G}_1)$ is connected.

  $(\because)$ $\forall G_1, G_2 \in \mathfrak{G}_1 \setminus \{\phi\}$, $G_1 \cap G_2 \neq \phi$.

- $(X, \mathfrak{G}_2)$ is not connected.

  $(\because)$ $\exists G_1 = \{a\}, G_2 = \{b, c\} \in \mathfrak{G}_2 \setminus \{\phi\}$:

  $$\begin{cases} G_1 \cap G_2 = \phi \\ X = G_1 \cup G_2. \end{cases}$$

ex

X with the trivial top.

i.e. $\mathfrak{G} = \{\phi, X\}$

Then, X is connected.

$(\because)$ $\forall G_1, G_2 \in \mathfrak{G} \setminus \{\phi\} = \{X\}$,

$$G_1 \cap G_2 = X \neq \phi.$$

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ex

$X = [0,1]$ $(\subset \mathbb{R})$

with the discrete top.

Then, X is not connected.

$(\because)$ Let $\begin{cases} G_1 = [0, \frac{1}{2}) \\ G_2 = [\frac{1}{2}, 1]. \end{cases}$

Then, $\begin{cases} \cdot\ G_1, G_2 \in \mathfrak{G} \setminus \{\phi\} \\ \cdot\ G_1 \cap G_2 = \phi \\ \cdot\ G_1 \cup G_2 = [0,1] \end{cases}$.

//

ex

(X, d) MS

$x, y \in X : x \neq y$

Then, $A = \{x, y\}$ is not connected.

$(\because)$ Let $\varepsilon = \frac{1}{3} d(x, y) > 0$.

Then,
- $S_\varepsilon(x), S_\varepsilon(y) \in \mathcal{G} \setminus \{\phi\}$.
- $S_\varepsilon(x) \cap A,\ S_\varepsilon(y) \cap A \neq \phi$
- $S_\varepsilon(x) \cap S_\varepsilon(y) = \phi$
- $A \subset S_\varepsilon(x) \cup S_\varepsilon(y)$.

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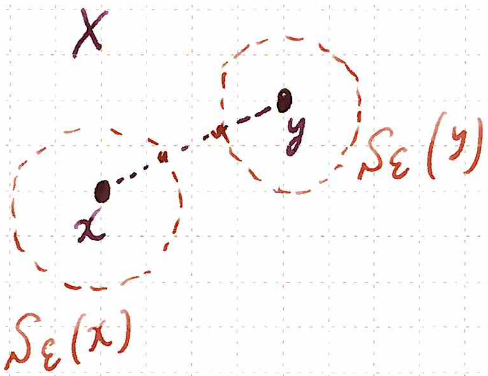

X set

$A, B \subset X$

Then, $B = A^c$

$$\iff \begin{cases} A \cap B = \phi \\ A \cup B = X \end{cases}$$

<u>Proof</u>

$(\Rightarrow)$ It obviously holds.

$(\Leftarrow)$ First, we show that $\underline{B \subset A^c}$.

Let $x \in B$.

As $A \cap B = \phi$, it holds that $x \notin A$.

$\therefore x \in A^c$.

Our next aim is to prove that

$\underline{B \supset A^c}$

Let $x \in A^c$. i.e. $x \notin A$

As $A \cup B = X$, we obtain $x \in B$.

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X set

$A \subset X$

Then, $A \subsetneq X \iff A^c \neq \emptyset$

$(X, \mathbb{G})$ top. space

$\Rightarrow$ Equivalent

① $(X, \mathbb{G})$ is not connected.

i.e. $\exists G_1, G_2 \in \mathbb{G} \setminus \{\emptyset\}$:
$$\begin{cases} G_1 \cap G_2 = \emptyset \\ X = G_1 \cup G_2 \end{cases}$$

② $\exists A \in \mathbb{G} \cap \mathbb{F} : \emptyset \subsetneqq A \subsetneqq X$

Proof

$$① \Leftrightarrow \exists A \in \mathbb{G} : \begin{cases} \emptyset \subsetneqq A \subsetneqq X \\ A^c \in \mathbb{G} \end{cases}$$

$$\Leftrightarrow \exists A \in \mathbb{G} : \begin{cases} \emptyset \subsetneqq A \subsetneqq X \\ A \in \mathbb{F} \end{cases}$$

$$\Leftrightarrow \exists A \in \mathbb{G} \cap \mathbb{F} : \emptyset \subsetneqq A \subsetneqq X .$$

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Cor

$(X, \mathfrak{S})$ top. space

$\Rightarrow$ Equivalent

① X is connected.

② ${}^{\forall}A \subset X$, $A \in \mathfrak{S} \cap \mathbb{F} \Rightarrow A = \emptyset$ or X

Connectedness (1)

1. 位相空間$(X, \boldsymbol{G})$において，その部分集合Aが連結ではないとはどう定義されるか？ 特に，$A = X$の場合についても述べよ．

2. 位相空間$(X, \boldsymbol{G})$において，その部分集合Aが連結であるとはどう定義されるか？ 特に，$A = X$の場合についても述べよ．

3. 次の$\mathbb{R}$の部分集合は連結か否か？ 理由とともに答えよ．
(1) $A = (0, 5]$ (2) $B = [0, 1) \cup \{3\}$ (3) $C = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$

4. 距離空間(X, d)の2つの異なる要素x, yを考え，$A = \{x, y\}$とすると，Aは連結ではない．このことを示せ．

5. どんな集合でも密着位相を入れるとそれは連結になる．なぜか？

6. 2つ以上の要素を含む集合に離散位相を入れると連結ではない．なぜか？

7. 集合$X = \{a, b, c, d, e\}$が連結になるような位相と連結にはならない位相を，それぞれ考えよ．

8. 位相空間$(X, \boldsymbol{G})$が連結でないということは，空集合と集合X以外に開集合かつ閉集合であるXの部分集合Aが存在することと同値である．このことを示せ．