Compactness (1)

Def.

$(X, \mathfrak{G})$ top. space

$A \subset X$

$\{G_\mu\} \subset \mathfrak{G}$: <u>open covering</u> of A.

$\Longleftrightarrow A \subset \bigcup_\mu G_\mu$

Def.

$(X, \mathfrak{G})$ top. space

X <u>compact</u>

$\iff {}^{\forall}\{G_\mu\} \subset \mathfrak{G} : X \subset \bigcup_\mu G_\mu,$

$\quad {}^{\exists}\{G_{\mu_i}\}_{i=1}^{N} \subset \{G_\mu\} : X \subset \bigcup_{i=1}^{N} G_{\mu_i}$

$(X, \mathfrak{G})$ is <u>not</u> compact.

$\iff {}^{\exists}\{G_\mu\} \subset \mathfrak{G} : X \subset \bigcup_\mu G_\mu,$

$\quad {}^{\forall}\{G_{\mu_i}\}_{i=1}^{N} \subset \{G_\mu\},\ X \not\subset \bigcup_{i=1}^{N} G_{\mu_i}$

- $(X, \mathcal{G})$ top. space
  $A = \{x_1, \cdots, x_N\} \subset X$
  $\Longrightarrow A$: compact

- $(X, \mathcal{G})$ top. space
  $\#\mathcal{G} < \infty$
  $A \subset X, \neq \phi$
  $\Longrightarrow A$: compact

- $(X, \mathcal{G})$ top. space
  with the trivial top.
  $A \subset X, \neq \phi$
  $\Longrightarrow A$: compact

ex

$\mathbb{R}$ is not compact.

Proof.

Let $I_n = (-n, n)$ $(n \in \mathbb{N})$.

Then, $\{I_n\}_{n\in\mathbb{N}}$ is an open covering of $\mathbb{R}$.

However, $\forall \{I_{n_i}\}_{i=1}^{N} \subset \{I_n\}_{n\in\mathbb{N}}$, $\mathbb{R} \not\subset \bigcup_{i=1}^{N} I_{n_i}$.

That is,

$\exists \{I_n\}_{n\in\mathbb{N}} \subset \mathfrak{G} : \mathbb{R} \subset \bigcup_{n\in\mathbb{N}} I_n :$

$\forall \{I_{n_i}\}_{i=1}^{N} \subset \{I_n\}_{n\in\mathbb{N}}$, $\mathbb{R} \not\subset \bigcup_{i=1}^{N} I_{n_i}$.

This shows that $\mathbb{R}$ is not compact.

//

ex

$(0, 1]$ $(\subset \mathbb{R})$ is not compact.

Proof

Let $G_n = \left(\frac{1}{n}, n+1\right)$ $(n \in \mathbb{N})$.

Then, $\{G_n\}_{n\in\mathbb{N}}$ is an open covering of $(0, 1]$.

However, $\forall \{G_{n_i}\}_{i=1}^{N} \subset \{G_n\}_{n\in\mathbb{N}}$,

$$(0, 1] \not\subset \bigcup_{i=1}^{N} G_{n_i}.$$

$\therefore$ $(0, 1]$ is not compact.

//

Th

$$\boxed{I = [a, b] (\subset \mathbb{R}) : \text{compact}}$$

Proof

Let $\{G_\mu\}_{\mu \in M}$ be an open covering of I.

1° Then, for $a \in I$, $\exists \mu_1 \in M : a \in G_{\mu_1}$.

If $I \subset G_{\mu_1}$, then the proof is completed.

Thus, suppose that $I \not\subset G_{\mu_1}$.

As G_{μ_1} is open in $\mathbb{R}$,

$\exists \alpha_1 \in (a, b] (\subset I) : [a, \alpha_1) \subset G_{\mu_1}$.

2° For $\alpha_1 \in I$, $\exists \mu_2 \in M : \alpha_1 \in G_{\mu_2}$.

If $I \subset G_{\mu_1} \cup G_{\mu_2}$, then

the proof is completed.

Therefore, suppose that $I \not\subset G_{\mu_1} \cup G_{\mu_2}$.

As G_{μ_2} is open in $\mathbb{R}$,

$\exists \alpha_2 \in (\alpha_1, b] (\subset I) : [a, \alpha_2) \subset G_{\mu_1} \cup G_{\mu_2}$.

......

Define $A \equiv \{\alpha \in (a, b] \mid [a, \alpha)$ is covered by a finite family of $\{G_\mu\}\}$.

As $\alpha_1, \alpha_2, \cdots \in A$, it holds that $A \neq \phi$.

Let $\bar{x} \equiv \sup A \in \mathbb{R}$.

We show that $\underline{\bar{x} = b}$.

Suppose for the sake of contradiction that

$\bar{x} < b$.

Then, $\exists \mu_0 \in M : \bar{x} \in G_{\mu_0}$.

As G_{μ_0} is open in $\mathbb{R}$,

$\exists \bar{x}' \in (\bar{x}, b] : [\bar{x}, \bar{x}') \subset G_{\mu_0}$.

Then, $[a, \bar{x}')$ is covered by

a finite family of $\{G_\mu\}$.

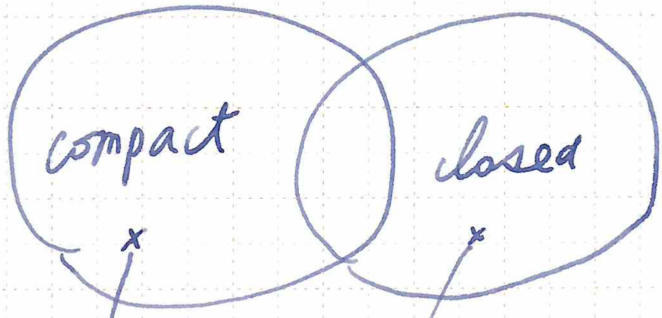
This contradicts the definition of $\bar{x}$. $\lrcorner$

For $b \in I$, $\exists \bar{\mu} \in M : b \in G_{\bar{\mu}}$.

Therefore, $I = [a, b]$ is covered

by a finite family of $\{G_\mu\}$. //

※ Heine-Borel theorem



- $A = X = \mathbb{R}$

  Then, A is closed in $\mathbb{R}$.

  However, A is not compact

- $A = X = \mathbb{R}$

  with the discrete metric

  Then, A is closed in $\mathbb{R}$.

  However, A is not compact.

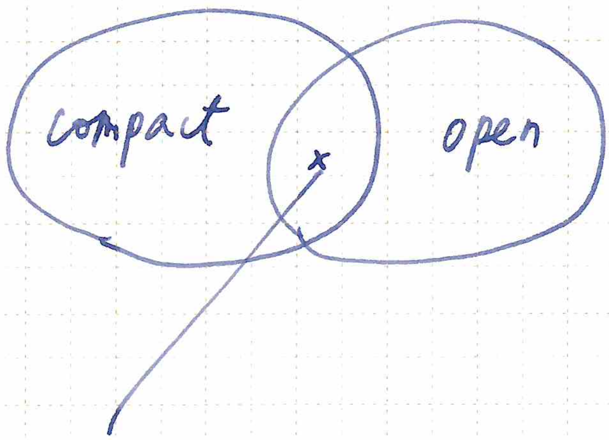
  (A is closed, open, and bdd.)

- $X = \mathbb{R}$ with the trivial top.

  $A = (0, 1)$

  Then, A is compact.

  However, A is not closed.



<u>ex</u>

$X = \{a, b, c\}$

$\mathfrak{G} = \{\emptyset, \{a\}, X\}$

Then, $(X, \mathfrak{G})$ is a top. space, and $\{a\}$ is compact and open in X.

$(X, \mathfrak{G})$ compact top. space
$A \subset X$ closed in X, $\neq \phi$
$\Rightarrow A$: compact

Proof

Let $\{G_\mu\}_{\mu \in M} (\subset \mathfrak{G})$ be an open covering of A.
We show that
$$\exists \{G_{\mu_i}\}_{i=1}^{N} \subset \{G_\mu\} : A \subset \bigcup_{i=1}^{N} G_{\mu_i}.$$

Note that $\{G_\mu\}_{\mu \in M}$ and A^c are a set of an open covering of X.

As X is compact,
$$\exists \{G_{\mu_i}\}_{i=1}^{N} \subset \{G_\mu\} : X = \left(\bigcup_{i=1}^{N} G_{\mu_i}\right) \cup A^c.$$

Therefore, $A \subset X = \left(\bigcup_{i=1}^{N} G_{\mu_i}\right) \cup A^c$.

This means that $A \subset \bigcup_{i=1}^{N} G_{\mu_i}$.

//

※ The inverse is not true.

$A \subset B \cup C$
$A \cap C = \phi$
$\Rightarrow A \subset B$

ex (without compactness of X)

$X = \mathbb{R}$: not compact

$A = [0, \infty)$: closed in X

However, A is not compact.

ex (without closedness of A)

$X = [-1, 1]$ $(\subset \mathbb{R})$ compact MS

$A = (0, 1) \subset X$: not closed in X

However, A is not compact.

ex ($\not\Leftarrow$)

$X = \{a, b, c\}$
$\mathcal{G} = \{\emptyset, X\}$
compact top. space

$A = \{a\}$ compact

However, A is not closed in $(X, \mathcal{G})$.

$(X, \mathfrak{G})$ top. space
$A, B \subset X$ compact
$\Rightarrow A \cup B$ compact

Proof

Let $\{G_\lambda\}_{\lambda \in \Lambda} \subset 2^X$ be an open covering of $A \cup B$.

Then,

$\exists \{G_{\lambda_\mu}\} \subset \{G_\lambda\}$: an open covering of A.

$\exists \{G_{\lambda_\nu}\} \subset \{G_\lambda\}$: 〃 of B.

As A is compact,

$$\exists \{G_{\lambda_{\mu_i}}\}_{i=1}^{M} \subset \{G_{\lambda_\mu}\} : A \subset \bigcup_{i=1}^{M} G_{\lambda_{\mu_i}}.$$

Similarly, as B is compact,

$$\exists \{G_{\lambda_{\nu_j}}\}_{j=1}^{N} \subset \{G_{\lambda_\nu}\} : B \subset \bigcup_{j=1}^{N} G_{\lambda_{\nu_j}}.$$

Consequently,

$$A \cup B \subset \left(\bigcup_{i=1}^{M} G_{\lambda_{\mu_i}} \right) \cup \left(\bigcup_{j=1}^{N} G_{\lambda_{\nu_j}} \right).$$

This implies that $A \cup B$ is compact.

//

X MS
$\{x_n\} \subset X : x_n \to x \in X$
$\Rightarrow A \equiv \{x_n \mid n \in \mathbb{N}\} \cup \{x\}$
is compact.

Proof

Let $\{G_\lambda\}$ be an open covering of A.

Then, $\exists \lambda_0 \in \Lambda : x \in G_{\lambda_0}$, which implies that $G_{\lambda_0} \in \mathcal{U}(x)$.

As $x_n \to x$,

$\exists n_0 \in \mathbb{N} : n \geq n_0 \Rightarrow x_n \in G_{\lambda_0}$.

Let $G_{\lambda_1} \in \{G_\lambda\} : x_1 \in G_{\lambda_1}, \cdots,$

$G_{\lambda_{n_0-1}} \in \{G_\lambda\} : x_{n_0-1} \in G_{\lambda_{n_0-1}}$.

Then, $\{G_{\lambda_i}\}_{i=0}^{n_0-1} \subset \{G_\lambda\}$

is an open covering of A.

Thus, A is compact. //

Def.

$X \neq \phi$

$\mathbb{A} \subset 2^X$

$\mathbb{A}$ has the finite intersection property (FIP)

$\Longleftrightarrow {}^\forall N \in \mathbb{N},\ A_1, \cdots, A_N \in \mathbb{A},\ \bigcap_{i=1}^{N} A_i \neq \phi$

<u>ex</u>

$\mathbb{R}$

$F_n = [n, \infty) \subset \mathbb{R}$

$\mathbb{F} = \{F_n \mid n \in \mathbb{N}\}$

Then, $\mathbb{F}$ has the FIP.

However, $\bigcap_{n \in \mathbb{N}} F_n = \phi$.

<u>ex</u>

$(0, 1]$

$F_n = (0, \frac{1}{n}] \subset (0, 1]$

closed in $(0,1]$

$\mathbb{F} = \{F_n \mid n \in \mathbb{N}\}$

Then, $\mathbb{F}$ has the FIP.

However, $\bigcap_{n \in \mathbb{N}} F_n = \phi$.

Th

> X top. space
>
> $\Rightarrow$ Equivalent
>
> ① X compact
>
> ② $\mathbb{F} = \{F_\mu \in 2^X \mid F_\mu : \text{closed in } X, \mu \in M\}$
>
> $\mathbb{F}$ has the FIP.
>
> $\Rightarrow \bigcap_{\mu \in M} F_\mu \neq \phi$

Proof

① $\Rightarrow$ ②

Assume by way of contradiction that

$$\bigcap_{\mu \in M} F_\mu = \phi.$$

Then, $X = \phi^c = \left(\bigcap_{\mu \in M} F_\mu\right)^c = \bigcup_{\mu \in M} F_\mu^c.$

This shows that $\{F_\mu^c\}_{\mu \in M}$ is an open covering of X.

From ①,

$$\exists \{F_{\mu_i}^c\}_{i=1}^N \subset \{F_\mu^c\}_{\mu \in M} : X = \bigcup_{i=1}^N F_{\mu_i}^c.$$

Hence, $\phi = X^c = \left(\bigcup_{i=1}^N F_{\mu_i}^c\right)^c = \bigcap_{i=1}^N F_{\mu_i}$

This contradicts that $\mathbb{F}$ has the FIP. $\lrcorner$

② ⇒ ①

Let $\{G_\lambda\}_{\lambda\in\Lambda} \subset 2^X$ be an open covering of X.

We show that

$$\exists \{G_{\lambda_i}\}_{i=1}^N \subset \{G_\lambda\} : X = \bigcup_{i=1}^N G_{\lambda_i}.$$

Define $F_\lambda = G_\lambda^c$.

Then, F_λ is closed in X $(\lambda \in \Lambda)$, and

$$X = \bigcup_{\lambda\in\Lambda} G_\lambda = \bigcup_{\lambda\in\Lambda} F_\lambda^c.$$

Therefore, we have

$$\phi = X^c = \left(\bigcup_{\lambda\in\Lambda} F_\lambda^c\right)^c = \bigcap_{\lambda\in\Lambda} F_\lambda.$$

From ②, $\{F_\lambda\}_{\lambda\in\Lambda}$ does not have the FIP,

that is,

$$\exists \{F_{\lambda_i}\}_{i=1}^N \subset \{F_\lambda\} : \bigcap_{i=1}^N F_{\lambda_i} = \phi.$$

Hence, $X = \phi^c = \left(\bigcap_{i=1}^N F_{\lambda_i}\right)^c$

$$= \bigcup_{i=1}^N F_{\lambda_i}^c = \bigcup_{i=1}^N G_{\lambda_i}$$

$\therefore$ $\exists \{G_{\lambda_i}\}_{i=1}^N = \{F_{\lambda_i}^c\}_{i=1}^N \subset \{F_\lambda^c\} = \{G_\lambda\}$:

$$X = \bigcup_{i=1}^N G_{\lambda_i}.$$

//

Th

$(X, \mathbb{G}), (Y, \mathbb{H})$ top. spaces
$A \subset X$ compact
$f: X \to Y$ continuous
$\Rightarrow f(A)\ (\subset Y)$: compact

Proof

Let $\{H_\mu\}$ be an open covering of $f(A)$.

i.e. ① $H_\mu \in \mathbb{H} \quad (\mu \in M)$
② $f(A) \subset \bigcup_{\mu \in M} H_\mu$

Using ②, we have

$$A \subset f^{-1}(f(A)) \subset f^{-1}\left(\bigcup_\mu H_\mu\right)$$
$$= \bigcup_\mu f^{-1}(H_\mu). \quad -(*)$$

As f is continuous, $f^{-1}(H_\mu) \in \mathbb{G}$.

Therefore, $(*)$ shows that

$\{f^{-1}(H_\mu)\}_{\mu \in M}$ is an open covering of A.

As A is compact,

$\exists \{H_{\mu_i}\}_{i=1}^N \subset \{H_\mu\}_{\mu \in M}$:

$$A \subset \bigcup_{i=1}^N f^{-1}(H_{\mu_i})$$

From this,

$$f(A) \subset f\left(\bigcup_{i=1}^{N} f^{-1}(H_{\mu_i})\right)$$

$$= \bigcup_{i=1}^{N} f\left(f^{-1}(H_{\mu_i})\right)$$

$$\subset \bigcup_{i=1}^{N} H_{\mu_i} .$$

$\therefore\ ^{\forall}\{H_\mu\}_{\mu \in M}$: open covering of $f(A)$,

$\exists \{H_{\mu_i}\}_{i=1}^{N} \subset \{H_\mu\}_{\mu \in M}$:

$$f(A) \subset \bigcup_{i=1}^{N} H_{\mu_i} .$$

$\therefore f(A)\ (\subset Y)$ is compact.

//

Compactness (1)

1. 位相空間がコンパクトであることとそうではないことの定義を述べよ.

2. 有限個しか要素を持たない集合は，どのような位相を入れてもコンパクトになる. なぜか？

3. どのような集合でも密着位相を入れるとコンパクトになる. なぜか？

4. 実数空間はコンパクトではない. 理由を述べよ.

5. 実数空間の部分空間$[-1,0)$はコンパクトではない. なぜか？

6. 実数空間内の有界閉区間$[a,b]$はコンパクトになる. このことを証明せよ.

7. コンパクトだが閉ではない集合と逆に閉集合だがコンパクトではない集合の例を挙げよ.

8. コンパクトな開集合の例を挙げよ.

9. コンパクトな位相空間$(X,\boldsymbol{G})$の非空閉部分集合Aはコンパクトである.
(1) このことを示せ.
(2) Xがコンパクトでないと，その非空閉部分集合はコンパクトとは限らない. このことを示す例を挙げよ.
(3) Xがコンパクトでも，その非空部分集合Aが閉でないとAはコンパクトになるとは限らない. このことを示す例を挙げよ.
(4) コンパクトな位相空間$(X,\boldsymbol{G})$のコンパクトな非空部分集合でも，閉集合になるとは限らない. このことを示す例を挙げよ.

10. 位相空間$(X,\boldsymbol{G})$の2つのコンパクトな部分集合A,Bについて，それらの合併集合$A\cup B$もコンパクトである. このことを示せ.

11. 一般の距離空間におけるコンパクト集合の例を挙げよ.

12. (**おまけ**) 位相空間$(X,\boldsymbol{G})$の部分集合族$\boldsymbol{A}$が有限交叉性を持つとはどういうことか？

13. (**おまけ**) 位相空間$(X,\boldsymbol{G})$がコンパクトであることは，Xの有限交叉性を持つ任意の閉部分集合族$\boldsymbol{F}$が共通点を持つことと同値である. このことを示せ.

14. (**おまけ**) 実数空間$\mathbb{R}$はコンパクトではないので，前問によると，有限交叉性を持つにもかかわらず共通点は持たないような閉部分集合族$\boldsymbol{F}$が存在する. そのような部分集合族の例を挙げよ. 実数空間$\mathbb{R}$のコンパクトではない部分集合$(0,1]$についても，同様な例を挙げよ.

15. $(X,\boldsymbol{G}_X),(Y,\boldsymbol{G}_Y)$を位相空間，$A$を$X$のコンパクトな部分集合とする. また，$f: X\to Y$を連続写像とする. このとき，$A$の$f$による像$f(A)$はコンパクトである. このことを証明せよ.