Connectedness (2)

Review

$(X, \mathfrak{G})$ top. space

$A \subset X$

- A is not connected.

$\Leftrightarrow \exists G_1, G_2 \in \mathfrak{G} \setminus \{\phi\}$:

$$\begin{cases} G_1 \cap A,\ G_2 \cap A \neq \phi \\ G_1 \cap G_2 = \phi \\ A \subset G_1 \cup G_2 \end{cases}$$

- X is not connected.

$\Leftrightarrow \exists G_1, G_2 \in \mathfrak{G} \setminus \{\phi\}$:

$$\begin{cases} G_1 \cap G_2 = \phi \\ X = G_1 \cup G_2 \end{cases}$$

- X is connected.

$\Leftrightarrow \forall G_1, G_2 \in \mathfrak{G} \setminus \{\phi\}$,

$G_1 \cap G_2 \neq \phi$ or $X \subsetneqq G_1 \cup G_2$

$$\boxed{\begin{array}{l} X, Y \neq \emptyset \\ f: X \to Y \\ H \subset Y \\ \text{Then, } \quad H \cap f(X) \neq \emptyset \\ \qquad\qquad \Leftrightarrow f^{-1}(H) \neq \emptyset \end{array}}$$

Proof.

It holds true that

$$\begin{aligned} & f^{-1}(H) \neq \emptyset \\ \Leftrightarrow\ & \exists x \in X : f(x) \in H \\ \Leftrightarrow\ & \exists y \in Y : y \in f(X) \cap H . \end{aligned}$$

//

Th

$(X, \mathfrak{G}_X), (Y, \mathfrak{G}_Y)$ top. spaces
$(X, \mathfrak{G}_X)$ connected
$f: X \to Y$ continuous
$\Rightarrow f(X) (\subset Y)$: connected

Proof

Suppose for the sake of contradiction that

$\exists H_1, H_2 \in \mathfrak{G}_Y \setminus \{\emptyset\}$:

$$\begin{cases} H_1 \cap f(X) \neq \emptyset, \quad H_2 \cap f(X) \neq \emptyset & \text{— ①} \\ H_1 \cap H_2 = \emptyset & \text{— ②} \\ f(X) \subset H_1 \cup H_2 & \text{— ③} \end{cases}$$

We prove that $\underline{(X, \mathfrak{G}_X) \text{ is not connected.}}$

i.e. $\exists G_1, G_2 \in \mathfrak{G}_X \setminus \{\emptyset\}$:

$$\begin{cases} G_1 \cap G_2 = \emptyset \\ X = G_1 \cup G_2 \end{cases}$$

Define $\begin{cases} G_1 = f^{-1}(H_1) \\ G_2 = f^{-1}(H_2) \end{cases}$

As $H_1, H_2 \in \mathfrak{G}_Y$ and f is continuous,

$G_1, G_2 \in \mathfrak{G}_X$.

From ①,

$$G_1\ (= f^{-1}(H_1)),\ G_2\ (= f^{-1}(H_2)) \neq \phi.$$

<u>$G_1 \cap G_2 = \phi$</u>

It holds true that

$$\begin{aligned} G_1 \cap G_2 &= f^{-1}(H_1) \cap f^{-1}(H_2) \\ &= f^{-1}(\underbrace{H_1 \cap H_2}_{= \phi \ ②}) \\ &= \phi. \end{aligned}$$

<u>$X = G_1 \cup G_2$</u>

We obtain

$$\begin{aligned} G_1 \cup G_2 &= f^{-1}(H_1) \cup f^{-1}(H_2) \\ &= f^{-1}(\underline{H_1 \cup H_2}) \quad ③ \\ &\supset f^{-1}(\underline{f(X)}) \\ &\supset X. \end{aligned}$$

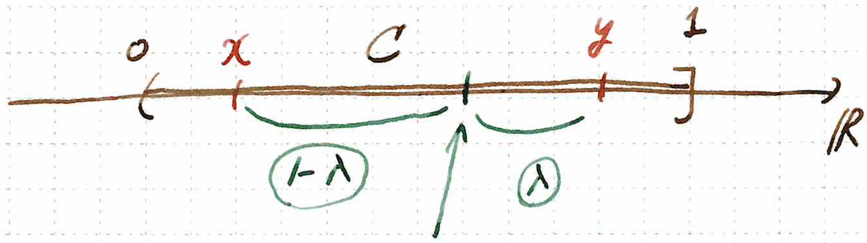
//

Def

$$C \subset \mathbb{R} \ \underline{\text{convex}}$$
$$\Leftrightarrow {}^\forall x, y \in C,\ \lambda \in (0,1),$$
$$\lambda x + (1-\lambda) y \in C$$

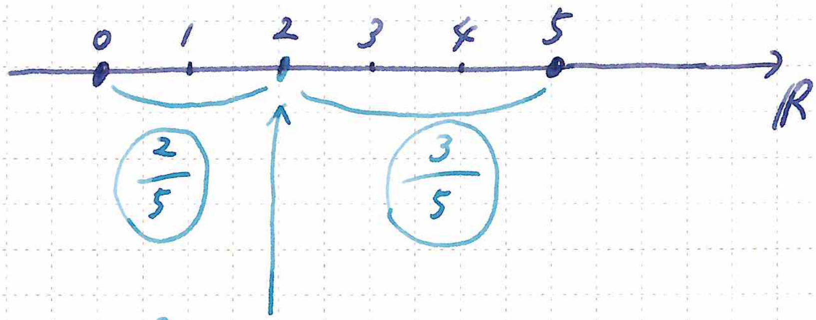
ex

$C = (0, 1]$ convex



$\lambda x + (1-\lambda) y \in C$

cf.



$$\frac{3}{5} \cdot 0 + \frac{2}{5} \cdot 5 = 2$$

※ $\emptyset$, $\{x\}$: convex.

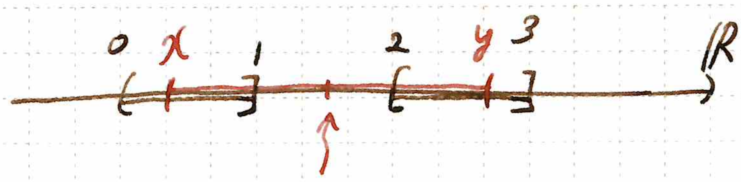
$C \ (\subset \mathbb{R})$ is not convex.

$\iff \exists x, y \in C, \ \lambda \in (0,1):$

$$\lambda x + (1-\lambda) y \notin C$$

ex

$(0,1] \cup [2,3]$: not convex



$\frac{1}{2}x + \frac{1}{2}y \notin C$

$C \subset \mathbb{R}, \neq \phi$

Then, C : convex

$\iff C$: interval

or

$C = \{x\}$ ← 一点集合

$\mathbb{R}^2$

$(x,y),(u,v)\in\mathbb{R}^2$

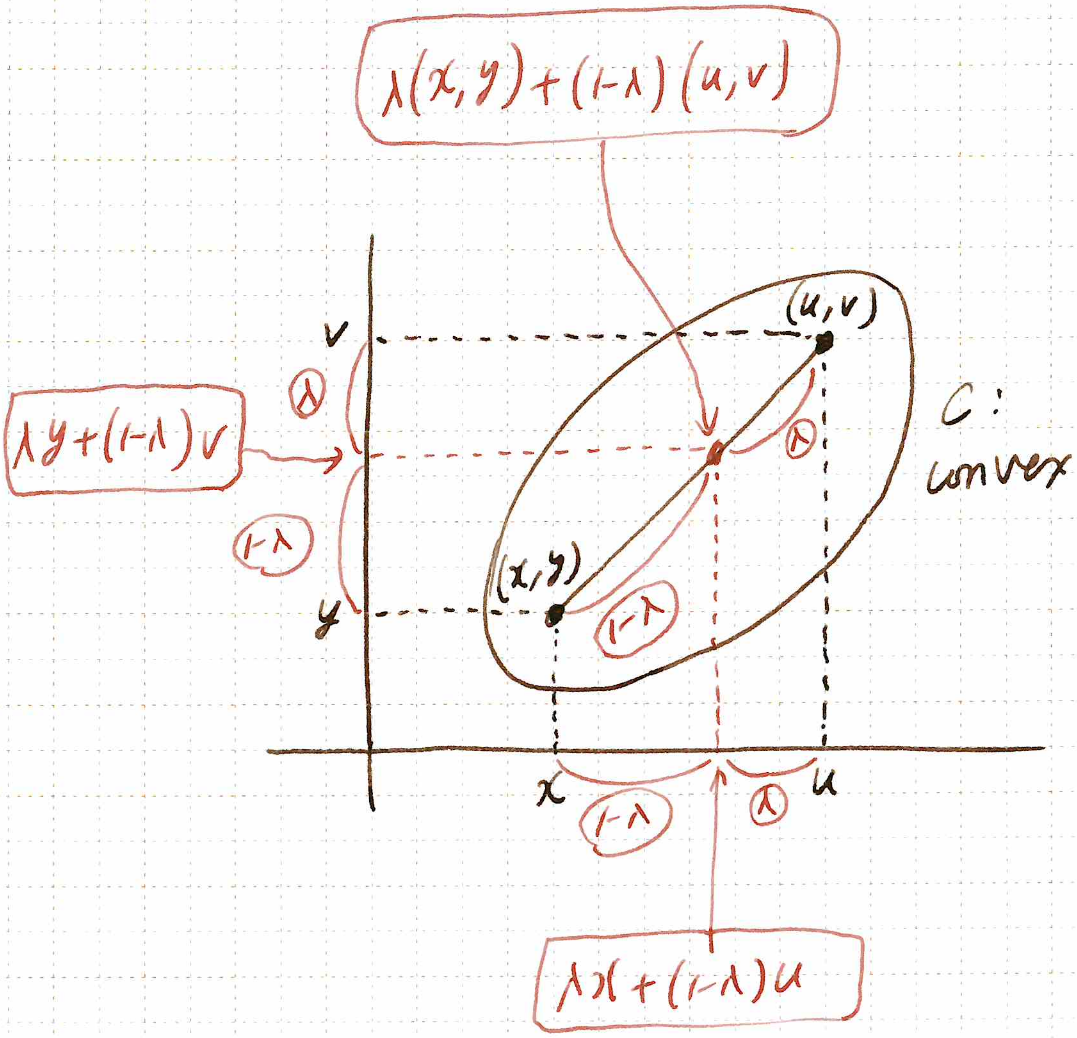
$\lambda\in\mathbb{R}$

- $(x,y)+(u,v)\equiv(x+u,y+v)$ 和
- $\lambda(x,y)\equiv(\lambda x,\lambda y)$ スカラー倍

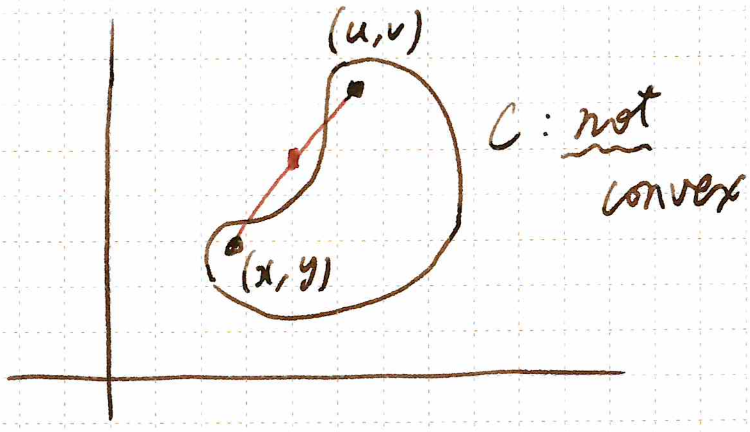
Def.

$$\boxed{\begin{array}{l} C\subset\mathbb{R}^2 \ \text{convex} \\ \Leftrightarrow {}^{\forall}(x,y),(u,v)\in C,\ \lambda\in(0,1), \\ \qquad \lambda(x,y)+(1-\lambda)(u,v)\in C \end{array}}$$

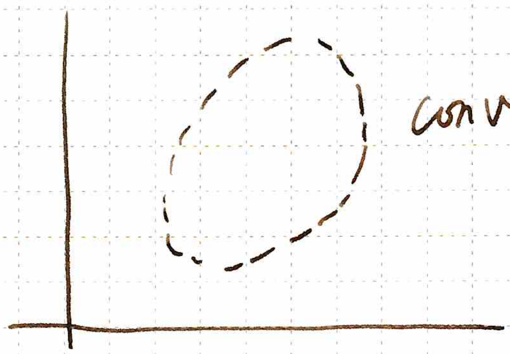
$$\begin{gathered} \parallel \\ (\lambda x,\lambda y)+((1-\lambda)u,(1-\lambda)v) \\ \parallel \\ (\lambda x+(1-\lambda)u,\ \lambda y+(1-\lambda)v) \end{gathered}$$



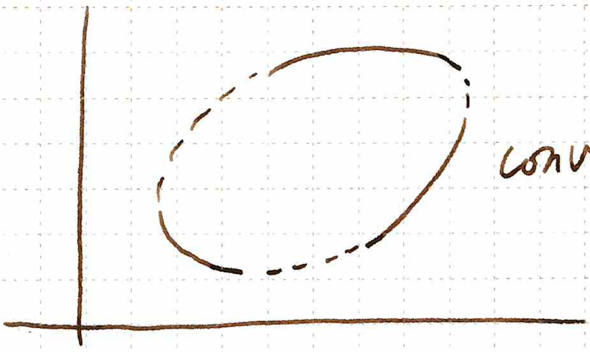
λ(x, y) + (1−λ)(u, v)
λy + (1−λ)v
v
y
λ
1−λ
(u, v)
(x, y)
C : convex
x
u
λx + (1−λ)u



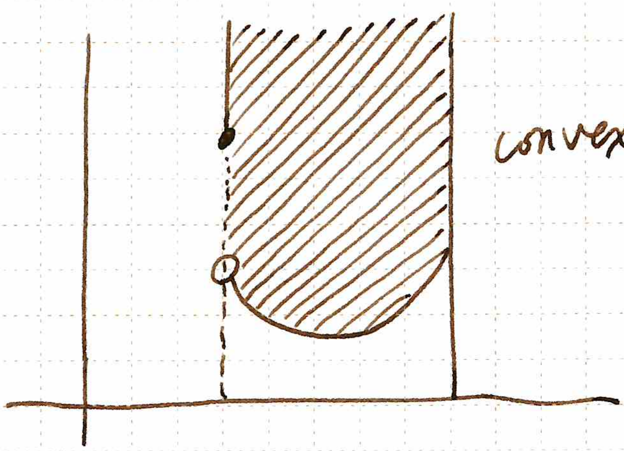
(u, v)
(x, y)
C : not convex



convex



convex



convex

$$\boxed{\begin{array}{l} C \subset \mathbb{R} \text{ connected} \\ \Rightarrow C : \text{convex} \end{array}}$$

<u>Proof</u>

Suppose to lead a contradiction that

C is not convex.

Then, $\exists x, y \in C, \lambda \in (0,1)$:

$$a \equiv \lambda x + (1-\lambda) y \notin C. \quad \text{—} (*)$$

We show that <u>C is not connected.</u>

i.e. $\exists G_1, G_2 \in \mathfrak{G} \setminus \{\phi\}$:

$$\left| \begin{array}{l} G_1 \cap C \neq \phi, \; G_2 \cap C \neq \phi \\ G_1 \cap G_2 = \phi \\ C \subset G_1 \cup G_2 \end{array} \right.$$

Let $G_1 = (-\infty, a)$ and $G_2 = (a, \infty)$.

$$\text{Then, } \left(\begin{array}{l} \cdot\; G_1, G_2 \in \mathfrak{G} \setminus \{\phi\} \\ \cdot\; G_1 \cap C \neq \phi, \; G_2 \cap C \neq \phi \\ \cdot\; G_1 \cap G_2 = \phi \end{array} \right.$$

<u>$C \subset G_1 \cup G_2$</u>

Let $z \in C$. From $(*)$, $z \neq a$.

As $G_1 \cup G_2 = (-\infty, a) \cup (a, \infty) = \mathbb{R} \setminus \{a\}$,

we obtain $z \in G_1 \cup G_2$.

$\therefore C \subset G_1 \cup G_2$.

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$C \subset \mathbb{R}$

→ Equivalent

① C : connected

② C : convex

Proof.

① ⇒ ② OK.

② ⇒ ①

Suppose for the sake of contradiction that C is not connected.

Then, $\exists G_1, G_2 \in \mathfrak{G} \setminus \{\phi\}$:

$$\begin{cases} G_1 \cap C \neq \phi, \quad G_2 \cap C \neq \phi & \text{—①} \\ G_1 \cap G_2 = \phi & \text{—②} \\ C \subset G_1 \cup G_2 & \text{—③} \end{cases}$$

From ①, we can choose

$$\begin{cases} a_0 \in G_1 \cap C \\ b_0 \in G_2 \cap C. \end{cases}$$

From ②, $a_0 \neq b_0$.

Assume, w.l.g., that $a_0 < b_0$.

Define $c_0 = \frac{1}{2} a_0 + \frac{1}{2} b_0$.

As $a_0, b_0 \in C$, it follows from ② that

$c_0 \in C$.

From ③, $c_0 \in G_1 \cup G_2$.

If $c_0 \in G_1$, then let

$$\begin{cases} a_1 = c_0 \in G_1 \cap C \\ b_1 = b_0 \in G_2 \cap C \end{cases}$$

If $c_0 \in G_2$, then let

$$\begin{cases} a_1 = a_0 \in G_1 \cap C \\ b_1 = c_0 \in G_2 \cap C \end{cases}$$

Define $c_1 = \frac{1}{2} a_1 + \frac{1}{2} b_1 \in C$.

From ③, $c_1 \in G_1 \cup G_2$.

If $c_1 \in G_1$, then let $\begin{cases} a_2 = c_1 \in G_1 \cap C, \\ b_2 = b_1 \in G_2 \cap C. \end{cases}$

If $c_1 \in G_2$, then let $\begin{cases} a_2 = a_1 \in G_1 \cap C, \\ b_2 = c_1 \in G_2 \cap C. \end{cases}$

$\cdots\cdots$

Repeating this operation, we obtain

$$[a_1, b_1] \supset [a_2, b_2] \supset \cdots$$

Note that $b_n - a_n = \left(\frac{1}{2}\right)^n (b-a) \to 0$.

As $\mathbb{R}$ is complete,

we have from Th (Cantor) that

$$\exists^1 \alpha \in \bigcap_{n=1}^{\infty} [a_n, b_n] \subset [a_0, b_0] \subset C.$$

From ③, $\alpha \in G_1 \cup G_2$.

Assume, w.l.g., that $\alpha \in G_1$.

As G_1 is open in $\mathbb{R}$,

$$\exists \varepsilon > 0 : (\alpha - \varepsilon, \alpha + \varepsilon) \subset G_1.$$

As $a_n \to \alpha$ and $b_n \to \alpha$, for $\varepsilon > 0$,

$$\exists n_0 \in \mathbb{N} : n \geq n_0 \Rightarrow a_n, b_n \in (\alpha - \varepsilon, \alpha + \varepsilon) \subset G_1.$$

As $b_n \in G_2$, this contradicts ②.

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Review

Th (Cantor)

X complete MS

$F_n \subset X \neq \phi$, closed $(n \in \mathbb{N})$

$F_1 \supset F_2 \supset \cdots$

$\delta(F_n) \equiv \sup\{d(x,y) \mid x, y \in F_n\} \to 0$

$\Rightarrow \exists 1 \; x^* \in \bigcap_{n=1}^{\infty} F_n$

Th

X top. space

$A \subset X$ connected

$f: A \to \mathbb{R}$ continuous

$x, y \in A$: $f(x) < f(y)$

$\Rightarrow {}^{\forall}\alpha \in [f(x), f(y)]$,

${}^{\exists}z \in A$: $f(z) = \alpha$

Proof

As A is connected and f is continuous,

$f(A)$ $(\subset \mathbb{R})$ is also connected.

Hence, $f(A)$ is an interval (convex).

As $f(x), f(y) \in f(A)$ with $f(x) < f(y)$,

it holds that $[f(x), f(y)] \subset f(A)$.

Consequently, the desired result holds.

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※ the intermediate value theorem

Cor

$I \subset \mathbb{R}$ interval

$f : I \to \mathbb{R}$ continuous

$x, y \in I : f(x) < f(y)$

$\Rightarrow {}^{\forall}\alpha \in [f(x), f(y)]$,

${}^{\exists}z \in I : f(z) = \alpha$

Proof

As $I (\subset \mathbb{R})$ is an interval,
it is connected.

Thus, from the previous theorem,
the desired result follows.

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Connectedness (2)

1. X, Yを空ではない集合，HをYの部分集合とする．写像$f : X \to Y$について，同値性

$$H \cap f(X) \neq \emptyset \Leftrightarrow f^{-1}(H) \neq \emptyset$$

を証明せよ．

2. 位相空間$(X, \boldsymbol{G}_X), (Y, \boldsymbol{G}_Y)$について，$(X, \boldsymbol{G}_X)$は連結とする．このとき，連続写像$f : X \to Y$による$X$の像$f(X)$が連結であることを示せ．また，逆は言えない（連続写像$f : X \to Y$によるXの像$f(X)$が連結であっても，Xが連結とは限らない）ことを示す例を考えよ．

3. 実数空間$\mathbb{R}$における空ではない凸集合は区間かまたは1点集合である．このことを納得せよ．

4. 平面$\mathbb{R}^2$における凸集合の定義を述べ，図を描いて説明せよ．

5. 実数空間$\mathbb{R}$においては，連結ならば凸である．このことを証明せよ．

6. 問題5の結果は，1次元空間$\mathbb{R}$でなければ言えない．2次元平面$\mathbb{R}^2$の場合，問題5における証明のどこがうまくいかなくなるか指摘するとともに，連結だが凸ではない集合の例を挙げよ．

7. 完備距離空間におけるCantorの定理を復習し，問題5の逆（凸ならば連結である）の証明をチェックせよ．（証明は少々長いので，自力でできなくてもかまわない．）

8.（中間値の定理の拡張）Aを位相空間Xの連結な部分集合とする．連続関数$f : A \to \mathbb{R}$と$x, y \in A$について，$f(x) < f(y)$とする．このとき，任意の$\alpha \in [f(x), f(y)]$について，ある$z \in A$が存在し$f(z) = \alpha$となる。このことを示せ．

9. 問題8から，位相空間Xとして特に$\mathbb{R}$とした場合の中間値の定理が導かれることを確認せよ．