Bolzano-Weierstrass theorem

X set

$\#X$ the <u>cardinality</u> of X

濃度

<u>ex</u>

$\#\{a\} = 1$

$\#\{a, b\} = 2$

$\#\{a, b, c\} = 3$

$\#\emptyset = 0$

<u>ex</u>

$\#\{1, 0, 1, 0, 1, 0, \cdots\} = 2$

$\#\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\} = \infty$

Th (Bolzano-Weierstrass)

$\{x_n\} \subset \mathbb{R}$ bdd

$\Rightarrow \exists \{x_{n_i}\} \subset \{x_n\}, x \in \mathbb{R} : x_{n_i} \to x$

Proof

(i) Assume that $\#\{x_n\} < \infty$.

Assume, w.l.g., that

$$\{x_n \mid n \in \mathbb{N}\} = \{a, b, c\}.$$

Let $\begin{cases} A = \{n \in \mathbb{N} \mid x_n = a\}, \\ B = \{n \in \mathbb{N} \mid x_n = b\}, \\ C = \{n \in \mathbb{N} \mid x_n = c\}. \end{cases}$

Then, $\#A = \infty$ or $\#B = \infty$ or $\#C = \infty$.

Assume, w.l.g., that $\#A = \infty$.

Let $\{x_{n_i}\} \subset \{x_n\} : \begin{cases} \forall i \in \mathbb{N}, x_{n_i} = a \\ n_1 < n_2 < \cdots \end{cases}$.

Then, $x_{n_i} \to a$. ⌟

(ii) Assume that $\#\{x_n\} = \infty$.

As $\{x_n\}$ is bdd,

$\exists [a,b] \subset \mathbb{R} : \forall n \in \mathbb{N},\ x_n \in [a,b]$.

1° Let $c = \frac{1}{2}(a+b) \in (a,b)$.

Then, $[a,c]$ or $[c,b]$ contains infinite elements of $\{x_n\}$.

Assume, w.l.g., that

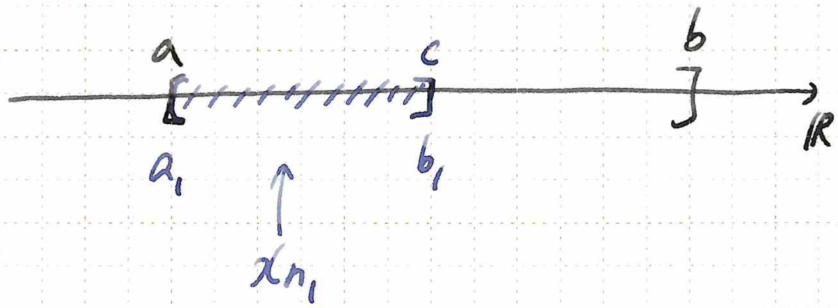
$$\#\{n \mid x_n \in [a,c]\} = \infty.$$

Define $\begin{cases} a_1 = a, \\ b_1 = c. \end{cases}$

i.e. $[a_1, b_1] = [a, c]$

Note that $|a_1 - b_1| = \frac{1}{2}|a-b|$.

Choose $x_{n_1} \in [a_1, b_1]$.



2° Let $c_1 = \frac{1}{2}(a_1 + b_1) \in (a_1, b_1)$.

Then, $[a_1, c_1]$ or $[c_1, b_1]$ contains infinite elements of $\{x_n\}$.

Assume, w.l.g., that

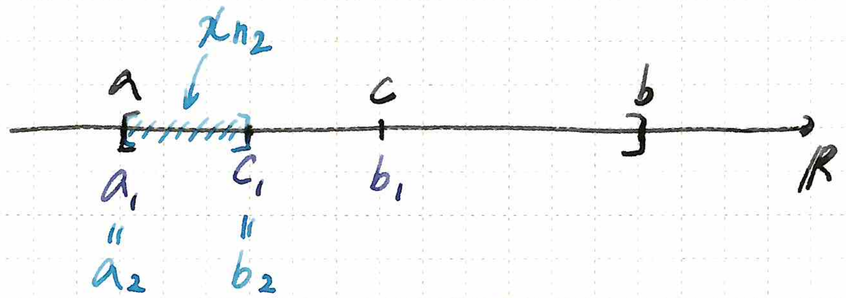
$$\#\{n \mid x_n \in [a_1, c_1]\} = \infty.$$

Define $\begin{cases} a_2 = a_1, \\ b_2 = c_1. \end{cases}$

i.e. $[a_2, b_2] = [a_1, c_1]$.

Note that $|a_2 - b_2| = \frac{1}{2^2}|a-b|$.

Choose $x_{n_2} \in [a_2, b_2]$, where $n_1 < n_2$.



3° Similarly, we can obtain $[a_3, b_3]$,

and $x_{n_3} \in [a_3, b_3]$, where $n_1 < n_2 < n_3$.

Using this method, we obtain a sequence of closed sets

$\{[a_i, b_i]\}$ s.t.

$$\begin{cases} [a_1, b_1] \supset [a_2, b_2] \supset \cdots \\ |a_i - b_i| = \frac{1}{2^i} |a-b| \end{cases} \quad \text{— (*)}$$

Furthermore,

$$\exists \{x_{n_i}\} \subset \{x_n\} : \ x_{n_i} \in [a_i, b_i] \ (\forall i \in \mathbb{N}) \quad \text{— (**)}$$

Note that

$$a \leqq a_1 \leqq a_2 \leqq \cdots\cdots \leqq b_2 \leqq b_1 \leqq b$$

That is,

- $\{a_n\}$ is $\begin{cases}\text{monotone increasing and} \\ \text{bdd above.}\end{cases}$
- $\{b_n\}$ is $\begin{cases}\text{monotone decreasing and} \\ \text{bdd below.}\end{cases}$

Consequently,

$$\begin{cases} \exists x_0 \in \mathbb{R} : a_n \to x_0 \\ \exists y_0 \in \mathbb{R} : b_n \to y_0 \end{cases} \quad (*3)$$

We prove that $\underline{x_0 = y_0}$.

From $(*)$ and $(*3)$, we have

$$|x_0 - y_0|$$

$$\leq \underbrace{|x_0 - a_n|}_{\to 0 \ (*3)} + \underbrace{|a_n - b_n|}_{\to 0 \ (*)} + \underbrace{|b_n - y_0|}_{\to 0}$$

$$\to 0.$$

From $(*3)$, $\begin{cases} a_n \to x_0 \\ b_n \to x_0. \end{cases}$

Using $(**)$, we obtain $x_{n_i} \to x_0$.

$\therefore \exists \{x_{n_i}\} \subset \{x_n\}, x_0 \in \mathbb{R} : x_{n_i} \to x_0$.

//

Th (Bolzano-Weierstrass)

$\{x_n\} \subset \mathbb{R}$ bdd

$\Rightarrow \exists \{x_{n_i}\} \subset \{x_n\}, x \in \mathbb{R} : x_{n_i} \to x$

(X, d) MS

$\{x_n\} \subset X$ bdd

$\not\Rightarrow \exists \{x_{n_i}\} \subset \{x_n\}, x \in X : x_{n_i} \to x$

counter ex.

$X = \mathbb{R}$ with the discrete metric

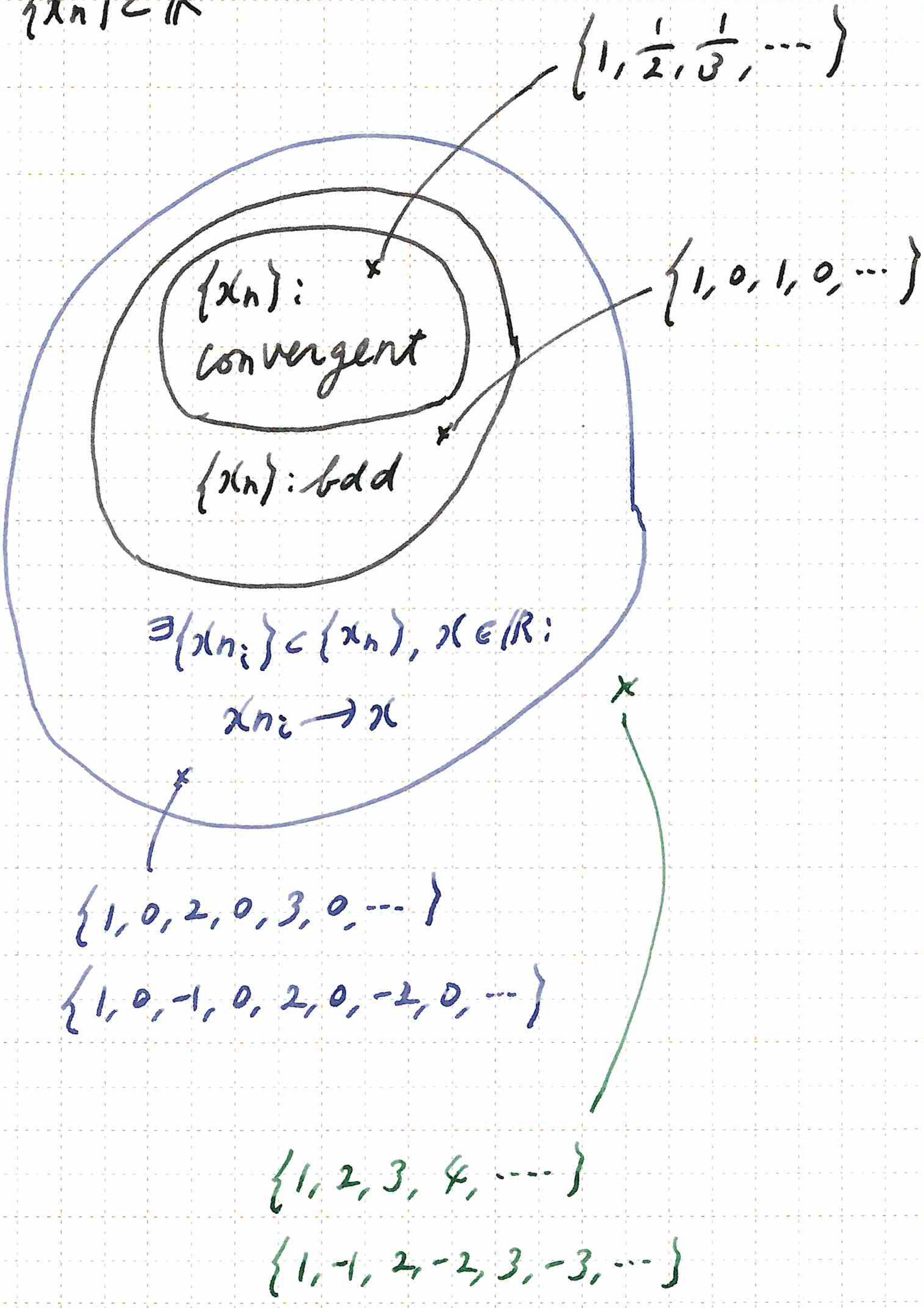
$\{x_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$

Then, $\{x_n\}$ is bdd.

However,

$\forall \{x_{n_i}\} \subset \{x_n\}, x \in \mathbb{R}, x_{n_i} \not\to x$

$\{x_n\} \subset \mathbb{R}$



Our next target is

Th.

$X \subset \mathbb{R} \neq \phi$, closed, bdd

$f : X \to \mathbb{R}$ continuous

$\Rightarrow$ f attains its maximum and minimum on X.

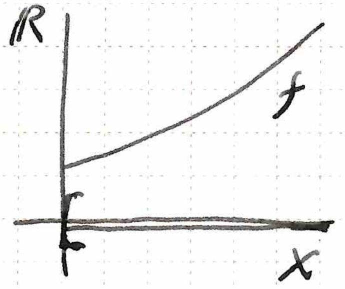
i.e. $\exists x^* \in X : f(x^*) = \sup_{x \in X} f(x) = \max_{x \in X} f(x)$

$\exists x_* \in X : f(x_*) = \inf_{x \in X} f(x) = \min_{x \in X} f(x)$

<extreme value theorem>

最大値・最小値の定理

ex

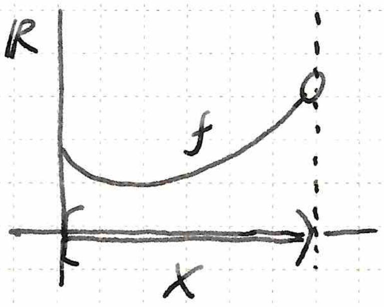


X is not bdd.

$\nexists x^* \in X:$

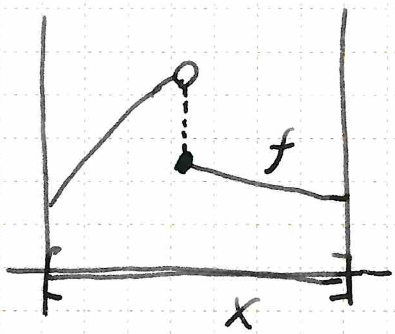
$f(x^*) = \sup_{x \in X} f(x)$

ex



X is not closed.

ex



f is not continuous.

Th

$X \subset \mathbb{R} \neq \emptyset$, closed

$\{x_n\} \subset X$ bdd

$\Rightarrow \exists \{x_{n_i}\} \subset \{x_n\}, x \in X : x_{n_i} \to x$

Proof

As $\{x_n\} (\subset X) \subset \mathbb{R}$ is bdd,

$\exists \{x_{n_i}\} \subset \{x_n\}, x \in \mathbb{R} : x_{n_i} \to x$.

As $\{x_{n_i}\} (\subset \{x_n\}) \subset X$, $x_{n_i} \to x \in \mathbb{R}$,

and X is closed in $\mathbb{R}$, we have

$x \in X$.

Thus, the desired result holds true. //

Cor

$X \subset \mathbb{R} \neq \emptyset$, closed, bdd

$\{x_n\} \subset X$

$\Rightarrow \exists \{x_{n_i}\} \subset \{x_n\}, x \in X : x_{n_i} \to x$

ex

$X = (0,1]$ bdd, not closed

$\{x_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\} \subset X$

Then,

$\nexists \{x_{n_i}\} \subset \{x_n\}, x \in X : x_{n_i} \to x$

i.e. $\forall \{x_{n_i}\} \subset \{x_n\}, x \in X, x_{n_i} \not\to x$

Lemma

$X \subset \mathbb{R}, \neq \emptyset$

$f: X \to \mathbb{R}$

$f(X) \, (\subset \mathbb{R})$ is not bdd above.

$\Rightarrow \exists \{x_n\} \subset X : f(x_n) \to \infty$

Proof

As $f(X) \, (\subset \mathbb{R})$ is not bdd above,

$\forall M > 0, \exists x \in X : f(x) > M.$

For $M = n \in \mathbb{N}$,

$\exists x_n \in X : f(x_n) > n.$

Thus, we have that

$\exists \{x_n\} \subset X : f(x_n) \to \infty.$

//

$$\boxed{\begin{array}{l} X \subset \mathbb{R} \ \neq \emptyset, \text{ closed, bdd} \\ f : X \to \mathbb{R} \text{ continuous} \\ \Rightarrow f(X) \ (\subset \mathbb{R}) : \text{bdd} \end{array}}$$

<u>Proof</u>

We show that <u>$f(X)$ is bdd above.</u>

Suppose for the sake of contradiction that

$f(X)$ is not bdd above.

Then, $\exists \{x_n\} \subset X : f(x_n) \to \infty$. — $(*)$

As $\{x_n\} \subset X$ and X is bdd and closed,

$\exists \{x_{n_i}\} \subset \{x_n\}, \ x \in X : x_{n_i} \to x$.

As f is continuous,

$$f(x) = f(\lim x_{n_i})$$
$$= \lim f(x_{n_i})$$

Thus, $\lim f(x_{n_i}) = f(x) \in \mathbb{R}$.

This contradicts $(*)$.

Hence, $f(X)$ is bdd above. $\lrcorner$

Similarly, we can prove that <u>$f(X)$ is bdd below.</u>

$\therefore f(X)$ is bdd.

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$X \subset \mathbb{R}$ $\neq \phi$, closed, bdd
$f: X \to \mathbb{R}$ continuous
$\Rightarrow f(X)\ (\subset \mathbb{R})$ is closed.

Proof

Let $\{y_n\} \subset f(X) : y_n \to y \in \mathbb{R}$.

i.e. $\forall n \in \mathbb{N}, \exists x_n \in X : y_n = f(x_n)$

We show that $y \in f(X)$.

i.e $\exists x \in X : y = f(x)$

As $\{x_n\} \subset X$ and X is closed bdd,

$\exists \{x_{n_i}\} \subset \{x_n\}, x \in X : x_{n_i} \to x$.

As $y_n \to y$, we have $y_{n_i} \to y$.

As f is continuous,

$$\begin{aligned} f(x) &= f\left(\lim_{i\to\infty} x_{n_i}\right) \\ &\overset{f:\text{ continuous}}{=} \lim_{i\to\infty} f(x_{n_i}) \\ &= \lim_{i\to\infty} y_{n_i} \\ &= \lim_{n\to\infty} y_n = y. \end{aligned}$$

Therefore, $\exists x \in X : y = f(x)$.

//

ex

$X = (0, 1)$ bdd, not closed

$f : X \to \mathbb{R}$ defined by

$f(x) = x \quad \forall x \in X$

Then, f is continuous and

$f(X) = (0, 1)$, which is not closed.

ex

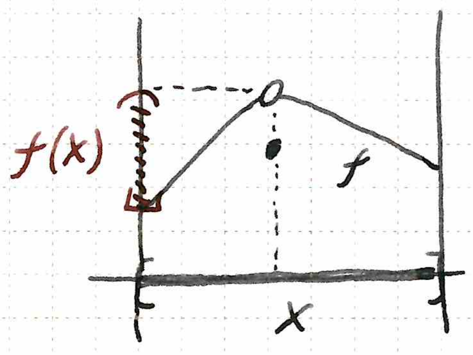
$X = (-\infty, 0]$ closed, not bdd

$f(x) = e^x$ continuous

Then, $f(X) = (0, 1]$,

which is not closed.

ex



$X \, (\subset \mathbb{R})$

bdd, closed

$f : X \to \mathbb{R}$

not continuous

$f(X)$ is not closed.

Th

$X \subset \mathbb{R}$ $\neq \emptyset$, closed, bdd

$f : X \to \mathbb{R}$ continuous

$$\Rightarrow \exists x^* \in X : f(x^*) = \sup_{x \in X} f(x)$$

$$\exists x_* \in X : f(x_*) = \inf_{x \in X} f(x)$$

Proof

We show that

$$\exists x^* \in X : f(x^*) = \sup_{x \in X} f(x) .$$

It holds that $f(X) (\subset \mathbb{R})$ is closed and bdd.

Thus, $\exists \alpha = \sup f(X) = \max f(X) \in f(X)$.

Consequently, $\exists x^* \in X : f(x^*) = \alpha = \max f(X)$.

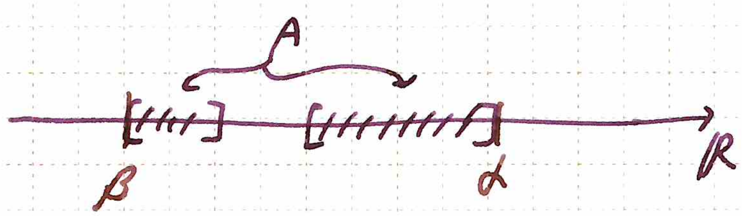
//

Review

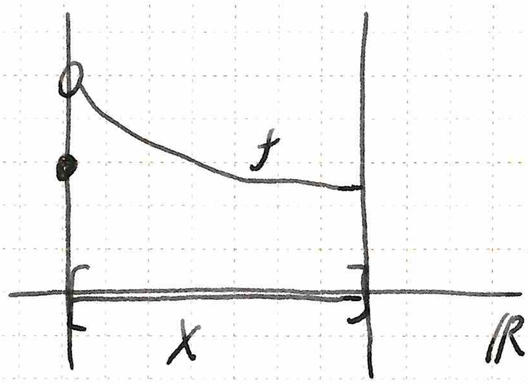
$A \subset \mathbb{R}$ $\neq \emptyset$, bdd, closed

$$\Rightarrow \exists \alpha = \max A = \sup A \in A$$

$$\exists \beta = \min A = \inf A \in A$$



ex



with the discrete metric

X : closed in $\mathbb{R}$, bdd

$f : X \to \mathbb{R}$ continuous

However,

$$\nexists\, x^* \in X : f(x^*) = \sup_{x \in X} f(x).$$

Th

$X \subset \mathbb{R}$ $\neq \emptyset$, closed, bdd
$f: X \to \mathbb{R}$ continuous
$\Rightarrow$ $^\exists x^* \in X : f(x^*) = \sup_{x \in X} f(x)$
$^\exists x_* \in X : f(x_*) = \inf_{x \in X} f(x)$

Proof

First, we show that <u>$f(X)$ $(\subset \mathbb{R})$ is bdd</u>.

Assume that $f(X)$ is not bdd above.

Then, $^\exists \{x_n\} \subset X : f(x_n) \to \infty$. —— (*)

As $\{x_n\} \subset X$ and X $(\subset \mathbb{R})$ is bdd,

$^\exists \{x_{n_i}\} \subset \{x_n\}, x \in \mathbb{R} : x_{n_i} \to x$.

As $\{x_{n_i}\} (\subset \{x_n\}) \subset X$, $x_{n_i} \to x$ and

X is closed in $\mathbb{R}$, we have $x \in X$.

As $f: X \to \mathbb{R}$, $f(x) \in \mathbb{R}$ exists.

As $x_{n_i} \to x$ and f is continuous,

$f(x_{n_i}) \to f(x) \in \mathbb{R}$ (as $i \to \infty$).

On the other hand, from (*), $f(x_{n_i}) \to \infty$.

This is a contradiction. ┘

Thus, $f(X)$ $(\subset \mathbb{R})$ is bdd above.

Similarly, $f(X)$ $(\subset \mathbb{R})$ is bdd below. ┘

As $f(X)$ $(\subset \mathbb{R})$ is bdd, $\exists \alpha = \sup f(X) \in \mathbb{R}$.

We prove that $\underline{\exists x^* \in X : f(x^*) = \alpha}$.

It holds that

$\exists \{z_n\} \subset X : f(z_n) \to \alpha$. — (**)

As $\{z_n\} \subset X$ and X is bdd,

$\exists \{z_{n_i}\} \subset \{z_n\}$ $(\subset X)$, $x^* \in \mathbb{R} : z_{n_i} \to x^*$.

As X is closed, $x^* \in X$.

$\therefore f(x^*) \in \mathbb{R}$ exists.

As f is continuous and $z_{n_i} \to x^*$,

we have that $f(z_{n_i}) \to f(x^*)$.

From (**),

$$\begin{aligned} \alpha &= \lim f(z_{n_i}) \quad \big) \; f: \text{continuous} \\ &= f(\lim z_{n_i}) \\ &= f(x^*) \end{aligned}$$

$\therefore \exists x^* \in X : f(x^*) = \alpha = \sup_{x \in X} f(x)$.

Considering $-f$, we obtain

$\exists x_* \in X : f(x_*) = \inf_{x \in X} f(x)$. //

※ an alternative proof

that directly uses Th (Bolzano-Weierstrass).

Bolzano–Weierstrass theorem

1. Bolzano–Weierstrassの定理の証明の概略を述べよ．また，証明の中で『上(下)に有界な単調増加(減少)数列は収束する』という結果が重要な役割を果たしていることを確認せよ．

2. 一般の距離空間においては，有界な点列でも収束する部分列を含むとは限らない．そのような距離空間と点列の例を挙げよ．

3. 実数空間において，収束する部分列を含むが有界ではない数列の例を挙げよ．(Bolzano–Weierstrassの定理の逆は言えない．)

4. Xを実数空間$\mathbb{R}$の有界閉部分集合，$\{x_n\}$をX内の点列とする．このとき，$\{x_n\}$の収束部分列が存在し，その部分列はX内に極限を持つ．Bolzano–Weierstrassの定理を用いて，この事実を示せ．

5. Xを実数空間$\mathbb{R}$の有界閉部分集合，$f : X \to \mathbb{R}$がを連続関数とする．関数fの像$f(X)(\subset \mathbb{R})$が有界集合であることを証明せよ．

6. Xを実数空間$\mathbb{R}$の有界閉部分集合，$f : X \to \mathbb{R}$がを連続関数とする．関数fの像$f(X)(\subset \mathbb{R})$が閉集合であることを証明せよ．

7. Xを実数空間$\mathbb{R}$の有界閉部分集合とする．連続関数$f : X \to \mathbb{R}$がX上で最大値と最小値を持つことを，問題4-6の結果を用いて証明せよ．

8. (おまけ)Xを実数空間$\mathbb{R}$の有界閉部分集合とする．連続関数$f : X \to \mathbb{R}$がX上で最大値と最小値を持つことを，問題4-6の結果を用いずBolzano–Weierstrassの定理を直接用いて証明せよ．

9. 問題7(および8)の最大値・最小値の定理は次の仮定に依拠している．すなわち，(a) Xの有界性，(b) Xの閉性，(c) fの連続性である．この3つの仮定がひとつでも満足されなくなると，定理の結論は保証されない．この3つの仮定(a)–(c)のうちひとつが満たされないために定理の結論が成り立たなくなる例を，3つの仮定それぞれについて挙げよ．