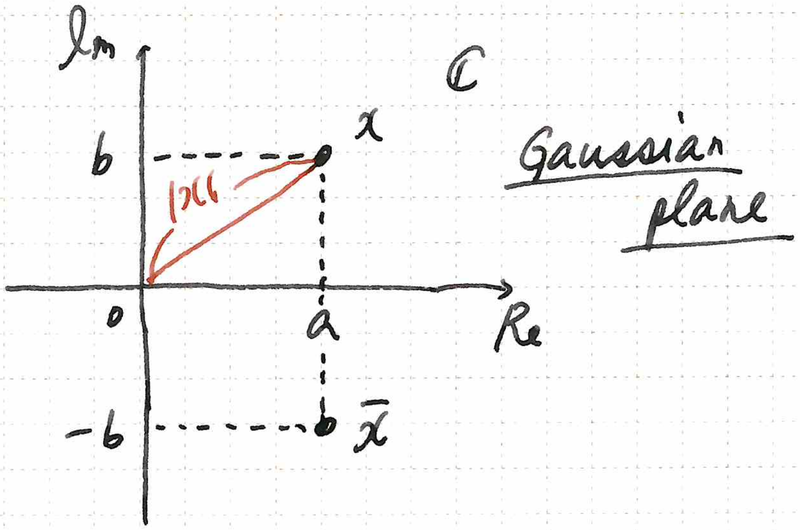
Vector spaces

complex numbers

Let $\begin{cases} x = a + bi \in \mathbb{C} \\ y = c + di \in \mathbb{C} \end{cases}$

where $a, b, c, d \in \mathbb{R}$



Def.

- $\bar{x} = a - bi \in \mathbb{C}$

  (complex conjugate)

  共役複素数

- $|x| = \sqrt{a^2 + b^2}$ absolute value

Definitions.

- $x+y = (a+bi)+(c+di)$
  $= (a+c)+(b+d)i$
- $x-y = (a-c)+(b-d)i$
- $xy = (a+bi)(c+di)$
  $= ac+adi+bci+bdi^2$
  $= (ac-bd)+(ad+bc)i$
- $$\frac{x}{y} = \frac{a+bi}{c+di}$$
  $$= \frac{(a+bi)(c-di)}{(c+di)(c-di)}$$
  $$= \frac{ac+bd-(ad-bc)i}{c^2+d^2}$$
  $$= \frac{ac+bd}{c^2+d^2} - \frac{ad-bc}{c^2+d^2}\,i$$

(分子・分母に $\bar{y}$ をかける.)

where $y \neq 0$ ($\Leftarrow c=d=0$).

Def

$V (\neq \phi)$ <u>K-vector space</u>

where $K = \mathbb{C}$ or $\mathbb{R}$

$+ : V \times V \to V$ (sum)

$\cdot : K \times V \to V$ (scalar multiplication)

(V1) $(x+y)+z = x+(y+z)$

(V2) $\exists 0 \in V : \forall x \in V,\ x+0 = x$

(V3) $\forall x \in V,\ \exists -x \in V : x+(-x) = 0$

(V4) $x+y = y+x$

(V5) $(\alpha\beta)x = \alpha(\beta x)$

(V6) $1 \cdot x = x$

(V7) $\alpha(x+y) = \alpha x + \alpha y$

(V8) $(\alpha+\beta)x = \alpha x + \beta x$

where $\left(\begin{array}{l} x, y, z \in V,\quad \alpha, \beta \in K \\ 1 = 1 + 0i \in \mathbb{C} \end{array} \right.$

From (V2) and (V4),

$$x + 0 = 0 + x = x.$$

From (V3) and (V4),

$$x + (-x) = (-x) + x = 0.$$

$$\boxed{(x+y)+(z+w) = (x+y+z)+w}$$

Proof

$$\begin{aligned} LHS &= x + (y + (z+w)) \\ &= x + ((y+z)+w) \\ &= (x+(y+z)) + w \\ &= RHS. \end{aligned}$$

//

ex

$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$

- $(x, y) + (u, v) = (x+u, y+v)$
- $\alpha(x, y) = (\alpha x, \alpha y)$

where $x, y, \alpha \in \mathbb{R}$

$\Rightarrow \mathbb{R}^2$: $\mathbb{R}$-vector space

"sum" defined on $\mathbb{R}^2$

$$(x, y) + (u, v) = (x+u, y+v)$$

sum on $\mathbb{R}$

equality on $\mathbb{R}^2$

ex

- $\lambda(x,y)+(1-\lambda)(u,v)$

$$=(\lambda x, \lambda y)+((1-\lambda)u, (1-\lambda)v)$$

$$=(\lambda x+(1-\lambda)u, \lambda y+(1-\lambda)v).$$

- $\alpha_1(x_1,y_1)+\alpha_2(x_2,y_2)+\alpha_3(x_3,y_3)$

$$=(\alpha_1 x_1, \alpha_1 y_1)+(\alpha_2 x_2, \alpha_2 y_2)$$

$$+(\alpha_3 x_3, \alpha_3 y_3)$$

$$=(\alpha_1 x_1+\alpha_2 x_2+\alpha_3 x_3, \alpha_1 y_1+\alpha_2 y_2+\alpha_3 y_3)$$

i.e. $\displaystyle\sum_{i=1}^{3}\alpha_i(x_i,y_i)=\left(\sum_{i=1}^{3}\alpha_i x_i, \sum_{i=1}^{3}\alpha_i y_i\right)$

In $\mathbb{R}^3$,

$$-2(1,0,-1)+3(1,-1,1)-2(-2,1,-2)$$

$$=(-2,0,2)+(3,-3,3)+(4,-2,4)$$

$$=(5,-5,9)$$

ex

$\mathbb{C}^2 = \{(x,y) \mid x, y \in \mathbb{C}\}$

- $(x,y)+(u,v)=(x+u,\, y+v)$
- $\alpha(x,y)=(\alpha x, \alpha y)$

$\Rightarrow \mathbb{C}^2$: $\mathbb{C}$-vector space

$\mathbb{R}$-vector space

Set of functions

$X \neq \emptyset$

$L(X) = \{ f \mid f : X \to \mathbb{R} \}$

$(f+g)(x) = f(x) + g(x) \quad \forall x \in X$

$(\alpha f)(x) = \alpha \cdot f(x) \quad \forall x \in X,\ \alpha \in \mathbb{R}$

$\Rightarrow L(X)$: $\mathbb{R}$-vector space

- $X = \mathbb{N}$

In this case,

$L(\mathbb{N}) = \{ x = \{x_n\} \mid x_n \in \mathbb{R}\ (n \in \mathbb{N}) \}$

- $\{x_n\} + \{y_n\} = \{x_n + y_n\}$
- $\alpha\{x_n\} = \{\alpha x_n\}$

- $X = \{1, 2\}$

In this case, $L(\{1,2\}) = \mathbb{R}^2$.

V vector space

$$x+0=x \quad \forall x\in V \qquad \text{①}$$

$$x+0'=x \quad \forall x\in V \qquad \text{②}$$

$$\Rightarrow 0=0' \ (\in V)$$

<u>Proof</u>

Setting $x=0'\ (\in V)$ in ①, we have

$$0'+0=0'.$$

Setting $x=0\ (\in V)$ in ②, we have

$$0+0'=0.$$

Using (V4), we have

$$0'=0'+0 \underset{(V4)}{=} 0+0'=0.$$

Consequently, we obtain $0=0'$. //

※ uniqueness of "zero element".

$$\boxed{\begin{array}{ll} V \text{ vector space} & \\ x \in V & \\ x+(-x)=0 & \text{— ①} \\ x+(-x)'=0 & \text{— ②} \\ \Rightarrow -x=(-x)' & \end{array}}$$

Proof.

It follows that

$$\begin{aligned}
-x &= -x+0 && \leftarrow (V2)\\
&= -x+\left(x+(-x)'\right) && \downarrow ②\\
&= (-x+x)+(-x)' && \downarrow (V1)\\
&= \left(x+(-x)\right)+(-x)' && \downarrow (V4)\\
&= 0+(-x)' && \downarrow (V2)\\
&= (-x)'+0 && \downarrow (V4)\\
&= (-x)'. && \downarrow (V2)
\end{aligned}$$

//

※ The inverse element is unique for all $x \in V$.

(1) $-(-x) = x$

(2) $x + y = z$

$\Rightarrow y = z + (-x)$

(3) $0 + 0 = 0$

(4) $-0 = 0$

(5) $x + x = x$

$\Rightarrow x = 0$

(6) $\underbrace{0}_{\in K} x = 0 \; (\in V)$

(7) $\alpha \cdot \underbrace{0}_{\in V} = 0 \; (\in V)$

Proof.

(1) As $x + (-x) = 0$, it holds that

$$(-x) + x = 0. \quad \leftarrow (V4)$$

This implies that x is an inverse element of $-x$.

As an inverse is unique, $-(-x) = x$. ⌟

(2) As $x + y = z$,

$$(x + y) + (-x) = z + (-x).$$

(V1)
(V4)

$$\therefore x + (-x) + y = z + (-x)$$

(V3)
(V4)(V2)

$$\therefore y = z + (-x).$$ ⌟

(3) From (v2), OK.

(4) From (3), OK.

(5) As $x + x = x$, we have

$$(x+x)+(-x) = x + (-x).$$

$$\therefore x + (x + (-x)) = x + (-x)$$

$$\therefore x + 0 = 0.$$

$$\therefore x = 0.$$

(6) It is sufficient to prove that

$$\underline{0x + 0x = 0x}.$$

$$\begin{aligned} LHS &= 0x + 0x \\ &\downarrow (v8) \\ &= (0+0)x \\ &= 0x. \end{aligned}$$

(7) We show that

$$\underline{\alpha 0 + \alpha 0 = \alpha 0}.$$

$$\begin{aligned} LHS &= \alpha 0 + \alpha 0 \\ &\downarrow (v7) \\ &= \alpha(0+0) \\ &\downarrow (3) \\ &= \alpha 0. \end{aligned}$$

//

$$\boxed{\begin{array}{l}\alpha x = 0 \\ \Longleftrightarrow \alpha = 0 \ (\in K) \text{ or } x = 0 \ (\in V)\end{array}}$$

Proof

$(\Leftarrow)$ OK

$(\Rightarrow)$

Assume that $\alpha \neq 0$.

It is sufficient to prove that

$$\underline{x = 0.}$$

This can be proved as follows:

$$
\begin{aligned}
x &= 1 \cdot x && \leftarrow (V6) \\
&= (\alpha^{-1}\alpha) x && \alpha \neq 0 \\
&= \alpha^{-1}(\alpha x) && (V5) \\
&= \alpha^{-1} 0 && (7) \\
&= 0.
\end{aligned}
$$

$\therefore x = 0.$ //

$$\boxed{(-1)x = -x}$$

Proof

We show that $x+(-1)x=0$.

It follows that

$$\begin{aligned} & x + (-1)x \\ &= 1\cdot x + (-1)x && (V6) \\ &= (1+(-1))x && (V8) \\ &= 0x \quad \text{where } 0 \in K \\ &= 0 \in V. && (6) \end{aligned}$$

//

We can write

$$x+(-y) = x-y.$$

· <u>linear combination</u>

$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_N x_N$

where $\alpha_i \in K$, $x_i \in V$ $(i = 1, \cdots, N)$

<u>Def.</u>

$x_1, \cdots, x_N \in V$

<u>linearly independent</u>

$\Leftrightarrow \alpha_1 x_1 + \cdots + \alpha_N x_N = 0 \ (\in V)$

$\Rightarrow \alpha_1 = \cdots = \alpha_N = 0 \ (\in K)$

$\downarrow$

$x_1, \cdots, x_N \in V$ <u>linearly dependent</u>

$\Leftrightarrow {}^{\exists}(\alpha_1, \cdots, \alpha_N) \in K^N$:

$$\begin{cases} (\alpha_1, \cdots, \alpha_N) \neq (0, \cdots, 0) \in K^N \\ \alpha_1 x_1 + \cdots + \alpha_N x_N = 0 \in V \end{cases}$$

ex

$\mathbb{R}^2$

$(1,0), (0,1)$: l.i.

(∵)

Assume that

$\alpha(1,0) + \beta(0,1) = (0,0)$

Then, $(\alpha, \beta) = (0,0)$.

$\therefore \alpha = \beta = 0$. //

ex

$\mathbb{R}^2$

$(1,1), (-2,-2)$: l.d.

(∵)

There exist $2, 1 \in \mathbb{R}$ s.t.

- $(2,1) \neq (0,0)$
- $2(1,1) + 1 \cdot (-2,-2) = (0,0) \in \mathbb{R}^2$.

//

ex

$$L(\mathbb{R}) = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$$
$$e^{ax}, e^{bx} \in L(\mathbb{R})$$
$$a, b \in \mathbb{R} : a \neq b$$
$$\Rightarrow e^{ax}, e^{bx} : \text{l.i.}$$

Proof.

Let $\alpha e^{ax} + \beta e^{bx} = 0 \ (\in L(\mathbb{R}))$. —(*)

We prove that $\alpha = \beta = 0 \ (\in \mathbb{R})$.

Substituting $x = 0$ in (*), we have

$$\alpha + \beta = 0. \quad \text{— ①}$$

Substituting $x = 1$ in (*), we have

$$\alpha e^{a} + \beta e^{b} = 0. \quad \text{— ②}$$

From ①, $\beta = -\alpha$. Using this, we have

$$\alpha e^{a} - \alpha e^{b} = 0.$$

$$\therefore \alpha(e^{a} - e^{b}) = 0.$$

As $a \neq b$, we obtain $\alpha = 0$.

From ①, $\beta = 0$. //

ex

$$\boxed{\begin{array}{l} L(\mathbb{N}\cup\{0\}) \\ \quad = \{x=\{x_n\} \mid x_n \in \mathbb{R}\ (n\in\mathbb{N}\cup\{0\})\} \\ a^n,\ b^n \in L(\mathbb{N}\cup\{0\}) \\ \text{where } a,b\in\mathbb{R},\ a,b\neq 0,\ a\neq b \\ \Rightarrow a^n,\ b^n \text{ are l.i.} \end{array}}$$

Proof.

Let $\alpha a^n + \beta b^n = 0 \ (\in L(\mathbb{N}\cup\{0\}))$. $-(*)$

We show that $\alpha = \beta = 0 \ (\in \mathbb{R})$.

Letting $n=0$ in $(*)$, we have

$$\alpha + \beta = 0. \quad -①$$

Letting $n=1$ in $(*)$, we have

$$\alpha a + \beta b = 0. \quad -②$$

From ① and ②, it follows that

$$\alpha a - \alpha b = 0.$$

$$\therefore \alpha(a-b) = 0.$$

As $a \neq b$, we obtain $\alpha = 0$.

From ①, $\beta = 0$. //

Def.

V K-vector space

$\dim V = \infty$

$\Leftrightarrow {}^{\forall}N \in \mathbb{N}, {}^{\exists}x_1, \cdots, x_N \in V$: l. i.

ex

$\dim L(\mathbb{R}) = \infty$

$\dim L(\mathbb{N}) = \infty$

V : K-vector space

- sum
- scalar multiplication

$\downarrow$

- 0 (zero element)
- linear combination

$\downarrow$

$x_1, \cdots, x_N$ $(\in V)$
linearly independent or not

$\downarrow$

$\boxed{\text{dimension}}$

Vector spaces

1. 二つの複素数$x = a + bi$, $y = c + di$について，両者の間の四則演算について，説明せよ．

2. 複素数$x = a + bi$の共役$\bar{x}$と絶対値$|x|$について，ガウス平面(複素平面)を描いて説明せよ．

3. ベクトル空間の定義を述べよ．また，実ベクトル空間の代表例として$\mathbb{R}^2$における和とスカラー倍の標準的な定義を述べ，ベクトル空間の定義に当てはまることを確認せよ．

4. 実ベクトル空間$\mathbb{R}^2$において，

$$\lambda(x,y) + (1-\lambda)(u,v) = (\lambda x + (1-\lambda)u,\ \lambda y + (1-\lambda)v)$$

が成り立つが，これを右辺から出発してそれが左辺と一致することを確認せよ．

5. 実ベクトル空間$\mathbb{R}^4$において，計算をフォローし空欄を埋めよ．

(1) $2(1,-2,2,3) - (3,1,-2,1) + 3(0,3,1,-1) = (\boxed{}, \boxed{}, \boxed{}, \boxed{})$

(2) $-3(1, \boxed{}, 2, 3) - 2(-3, 1, -2, \boxed{}) + 3(0, -1, \boxed{}, -1) = (\boxed{}, -5, 4, -10)$

6. ベクトル空間において，零元と逆元の一意性を証明せよ．

7. ベクトル空間Vにおいて，次を示せ．ただし，$x, y, z \in V$, αはスカラーである．

(1) $-(-x) = x$,　(2) $x + y = z$ならば，$y = z + (-x)$,　(3) $0 + 0 = 0$,　(4) $-0 = 0$,

(5) $x + x = x$ならば，$x = 0$,　(6) $0x = 0$,　(7) $\alpha 0 = 0$.

8. ベクトル空間において，次を示せ．

(1) $\alpha x = 0 \Leftrightarrow \alpha = 0$ or $x = 0$,

(2) $(-1)x = -x$.

9. 実ベクトル空間$\mathbb{R}^2$において，$(1,2)$と$(-1,-1)$は一次独立である．このことを示せ．

10. ベクトル空間Vにおいて，$x, \lambda x (\in V)$は一次従属である．このことを示せ．ただし，ここで，λはスカラーである．

11. 実ベクトル空間$L(\mathbb{R}) = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$において，$x, x^2, x^3$は一次独立であることを証明せよ．

12. 実ベクトル空間$L(\mathbb{R}) = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$は無限次元である．なぜか？

解答. 5. (1) $(-1, 4, 9, 2)$,　(2) (順に) $0, 2, -1, 3$.