The nearest point theorem and metric projections

Th

> H Hilbert space over $\mathbb{R}$
> $C \subset H \neq \emptyset$, closed, convex
> $\Rightarrow \forall x \in H, \exists^1 y_0 \in C : \|x - y_0\| = d(x, C)$

Proof

Let $x \in H$ and

define $d \equiv d(x, C) \equiv \inf_{y \in C} \|x - y\|$. $-(*)$

⟨Existence⟩

From $(*)$, $\exists \{y_n\} \subset C : \|x - y_n\| \to d$. $-(**)$

We show that $\{y_n\}$ is a Cauchy sequence.

Let $m, n \in \mathbb{N} : m \geq n$.

As $\{y_n\} \subset C$ and C is convex,

$$\lambda y_m + (1-\lambda) y_n \in C \quad \forall \lambda \in (0,1).$$

Therefore,

$$\begin{aligned} d^2 &\leq \|x - [\lambda y_m + (1-\lambda) y_n]\|^2 \\ &= \|\lambda (x - y_m) + (1-\lambda)(x - y_n)\|^2 \\ &= \lambda \|x - y_m\|^2 + (1-\lambda) \|x - y_n\|^2 - \lambda(1-\lambda)\|y_m - y_n\|^2 \end{aligned}$$

$$\begin{aligned} \therefore \lambda(1-\lambda)\|y_m - y_n\|^2 &\leq \lambda \|x - y_m\|^2 + (1-\lambda)\|x - y_n\|^2 - d^2 \\ &\to 0 \text{ as } m, n \to \infty. \end{aligned}$$

As H is complete and $C(\subset H)$ is closed,
C is also complete.

Thus, $\exists y_0 \in C : y_n \to y_0$.

Observe that $\underline{\|x-y_0\| = d}$.

Indeed, $d \underset{(*)}{=} \lim \|x-y_n\|$

$= \|\lim (x-y_n)\|$

$= \|x-y_0\|$.

<Uniqueness>

Let $y_0, z_0 \in C : d = \|x-y_0\| = \|x-z_0\|$.

As C is convex, $\lambda y_0 + (1-\lambda) z_0 \in C \quad {}^\forall \lambda \in (0,1)$

It follows that

$$
\begin{aligned}
d^2 &\leq \|x - [\lambda y_0 + (1-\lambda) z_0]\|^2 \\
&= \|\lambda(x-y_0) + (1-\lambda)(x-z_0)\|^2 \\
&= \lambda \|x-y_0\|^2 + (1-\lambda)\|x-z_0\|^2 - \lambda(1-\lambda)\|y_0-z_0\|^2 \\
&= d^2 - \lambda(1-\lambda)\|y_0-z_0\|^2.
\end{aligned}
$$

$\therefore \lambda(1-\lambda)\|y_0-z_0\|^2 \leq 0$.

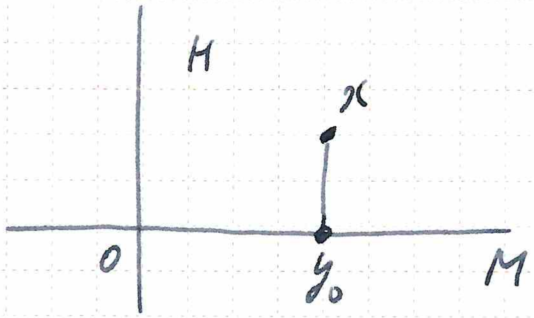
$\therefore y_0 = z_0$. //

Cor

H Hilbert space over $\mathbb{R}$
$C \subset H$ compact, convex
$\Rightarrow \forall_{x \in H}, \exists^1 y_0 \in C : \|x - y_0\| = d(x, C)$

Cor

H Hilbert space over $\mathbb{R}$
$M \subset H$ closed subspace
$\Rightarrow \forall_{x \in H}, \exists^1 y_0 \in M : \|x - y_0\| = d(x, M)$

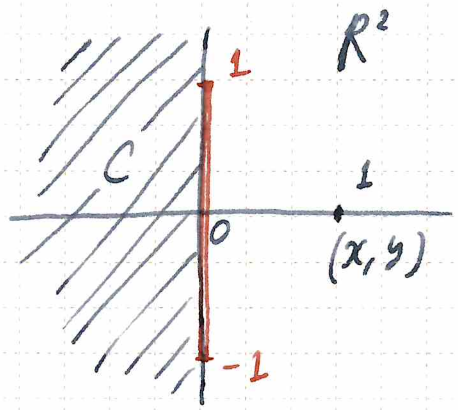


<u>ex</u>

$(\mathbb{R}^2, \|\cdot\|_\infty)$

$C = \{(u,v) \mid u \leq 0\}$

$(x,y) = (1,0)$

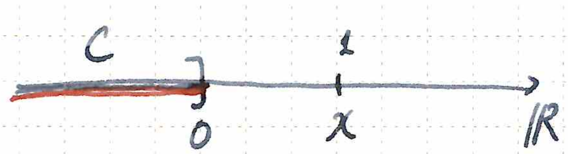


<u>ex</u>

$\mathbb{R}$ with the discrete metric

$C = (-\infty, 0] \neq \emptyset$, closed, convex

$x = 1$



H $\mathbb{R}$-pre-Hilbert space

$C \subset H$, $\neq \emptyset$

$x, \bar{x} \in H$

$\langle x-\bar{x}, \bar{x}-y \rangle \geqq 0 \quad {}^{\forall} y \in C$ — (*)

$\Rightarrow \|x-\bar{x}\| \leqq \|x-y\| \quad {}^{\forall} y \in C$

Proof

Let $y \in C$.

We prove that $\underline{\|x-\bar{x}\| \leqq \|x-y\|.}$

From (*), $\langle x-\bar{x}, \bar{x}-y \rangle \geqq 0$.

$\therefore \langle x-\bar{x}, \bar{x}-x+x-y \rangle \geqq 0$

$\therefore \langle x-\bar{x}, \bar{x}-x \rangle + \langle x-\bar{x}, x-y \rangle \geqq 0$.

$$\begin{aligned}\therefore \|x-\bar{x}\|^2 &\leqq \langle x-\bar{x}, x-y \rangle \\ &\leqq \|x-\bar{x}\| \|x-y\|.\end{aligned}$$

We obtain $\|x-\bar{x}\| \leqq \|x-y\| \quad {}^{\forall} y \in C$.

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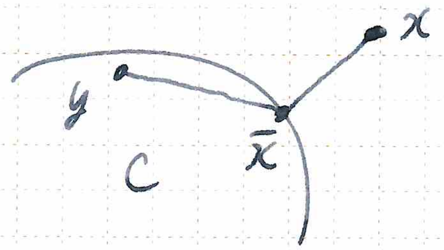


In this case,

$$\langle x-\bar{x}, \bar{x}-y \rangle \geqq 0 \quad \forall y \in C$$

ex

$H = \mathbb{R}$

$C = (-\infty, 0]$

$x = 2$

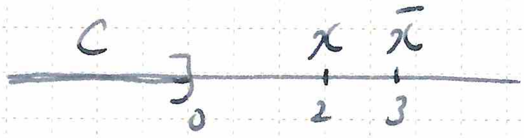
$\bar{x} = 3$

The reverse is not always true.

Then, $\|x-\bar{x}\| \leqq \|x-y\| \quad \forall y \in C$

However, $\langle x-\bar{x}, \bar{x}-y \rangle$

$$= (x-\bar{x})(\bar{x}-y) < 0.$$



Th

H $\mathbb{R}$-pre-Hilbert space

$C \subset H$ $\neq \emptyset$, convex

$x \in H$, $\bar{x} \in C$

$\Rightarrow$ Equivalent

① $\|x-\bar{x}\| \leq \|x-y\|$ $\forall y \in C$

② $\langle x-\bar{x}, \bar{x}-y \rangle \geq 0$ $\forall y \in C$

③ $\|x-\bar{x}\|^2 + \|\bar{x}-y\|^2 \leq \|x-y\|^2$ $\forall y \in C$

Proof

①⇒②

Let $y \in C$.

As $\bar{x}, y \in C$ and C is convex,

$$\lambda \bar{x} + (1-\lambda) y \in C \quad \forall \lambda \in (0,1).$$

From ①,

$$\begin{aligned}
\|x-\bar{x}\|^2 &\leq \|x-[\lambda \bar{x} + (1-\lambda) y]\|^2 \\
&= \|x-[(\lambda-1)\bar{x} + \bar{x} + (1-\lambda) y]\|^2 \\
&= \|x-\bar{x} + (1-\lambda)(\bar{x}-y)\|^2 \\
&= \|x-\bar{x}\|^2 + 2(1-\lambda)\langle x-\bar{x}, \bar{x}-y \rangle \\
&\quad + (1-\lambda)^2 \|\bar{x}-y\|^2
\end{aligned}$$

As $1-\lambda > 0$,

$$-(1-\lambda)\|\bar{x}-y\|^2 \leq 2\langle x-\bar{x}, \bar{x}-y \rangle \quad \forall \lambda \in (0,1).$$

As $\lambda \uparrow 1$, $0 \leq \langle x-\bar{x}, \bar{x}-y \rangle$ $\forall y \in C$.

② ⟺ ③

It follows that

$$② \iff 2\langle x-\bar{x}, \bar{x}-y\rangle \geqq 0 \quad \forall y \in C$$

$$\iff \|x-y\|^2+\|\bar{x}-\bar{x}\|^2 - \|x-\bar{x}\|^2-\|\bar{x}-y\|^2 \geqq 0 \quad \forall y \in C$$

$$\iff \|x-\bar{x}\|^2+\|\bar{x}-y\|^2 \leqq \|x-y\|^2 \quad \forall y \in C.$$

$$\iff ③.$$

③ ⟹ ①

OK.

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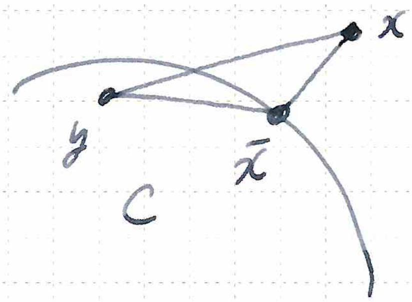


Def

H Hilbert space over $\mathbb{R}$

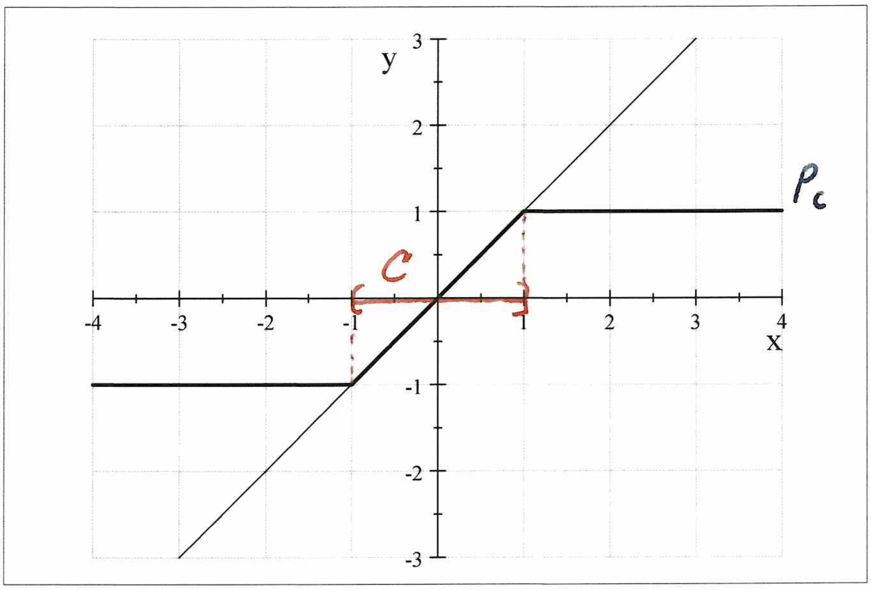
$C \subset H \neq \emptyset$, closed, convex

$P_C : H \to C$ <u>metric projection</u> onto C

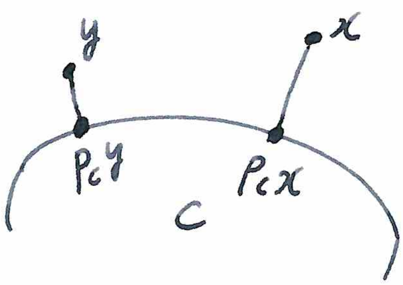
$$P_C : x \longmapsto P_C x \quad \text{s.t.}$$

$$\|x - P_C x\| \leq \|x - y\| \quad \forall y \in C$$

ex
$H = \mathbb{R}$
$C = [-1, 1]$
Then,
$$P_C x = \begin{cases} -1 & x < -1 \\ x & -1 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$



ex
$H = \mathbb{R}^2$



Def

H $\mathbb{R}$-pre-Hilbert space

$C \subset H$, $\neq \phi$

$T : C \to H$ monotone

$\iff$ $^{\forall} x, y \in C$, $\langle x-y, Tx-Ty \rangle \geqq 0$

Remark.

$C \subset \mathbb{R}$, $\neq \phi$

$T : C \to \mathbb{R}$

$\Rightarrow$ Equivalent

① T: monotone

② T: monotone increasing

i.e. $x \leqq y \Rightarrow Tx \leqq Ty$

H $\mathbb{R}$-pre-Hilbert space
$M \subset H$ subspace
$T: M \to H$ linear
$\Longrightarrow$ Equivalent
① T: monotone
② $^\forall x \in M$, $\langle Tx, x \rangle \geqq 0$

Proof

①⇒②

Let $x \in M$.
From ①, $\langle x-0, Tx-T0 \rangle \geqq 0$.
$\therefore \langle x, Tx \rangle \geqq 0$.

②⇒①

Let $x, y \in M$.
It follows that

$$
\begin{aligned}
&\langle x-y, Tx-Ty \rangle \\
&\quad = \langle x-y, T(x-y) \rangle \quad (T: \text{linear}) \\
&\quad \geqq 0 \quad (②)
\end{aligned}
$$

//

※ ②：行列の場合の、半正定値性.

Th

$\boxed{\begin{array}{l} H \ \mathbb{R}\text{-Hilbert space} \\ C \subset H \neq \emptyset, \text{closed, convex} \\ P_C : H \to C \ \text{MP} \\ \Rightarrow P_C : \text{NE, monotone} \end{array}}$

Proof

Let $x, y \in H$.

We show that

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle. \quad -(*)$$

It follows that

$$\langle x - P_C x, P_C x - P_C y \rangle \geq 0, \quad -①$$

$$\langle x - P_C y, P_C y - P_C x \rangle \geq 0. \quad -②$$

From ②, $\langle -y + P_C y, P_C x - P_C y \rangle \geq 0. \quad -②'$

① + ②' yields

$$\langle x - y - (P_C x - P_C y), P_C x - P_C y \rangle \geq 0.$$

Thus, we have

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle. \quad \lrcorner$$

(*) directly indicates that

$$0 \leq \|P_C x - P_C y\|^2 \leq \langle x-y, P_C x - P_C y \rangle.$$

Thus, P_C is monotone.

Furthermore, from (*),

$$\|P_C x - P_C y\|^2 \leq \langle x-y, P_C x - P_C y \rangle$$
$$\leq \|x-y\| \|P_C x - P_C y\|.$$

This means that

$$\|P_C x - P_C y\| \leq \|x-y\|.$$

Hence, P_C is NE.

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Def

$T: C \to H$ firmly nonexpansive

$\Leftrightarrow$ $\forall x, y \in C$,

$$\|Tx - Ty\|^2 \leq \langle x-y, Tx-Ty \rangle$$

$C \subset \mathbb{R}$, $\neq \emptyset$
$T : C \to \mathbb{R}$
$\Rightarrow$ Equivalent

① T: firmly NE
i.e. $(Tx - Ty)^2 \leq (x-y)(Tx - Ty)$

② T: monotone increasing, NE

Proof

① $\Rightarrow$ ② OK.

② $\Rightarrow$ ①

Let $x, y \in C (\subset \mathbb{R})$.
Assume, w.l.g., that $y < x$.
As T is monotone increasing, we have that $Ty \leq Tx$.
As T is NE, $|Tx - Ty| \leq |x - y|$.
Hence, $Tx - Ty \leq x - y$.
Multiplying $Tx - Ty \,(\geq 0)$, we obtain

$$(Tx - Ty)^2 \leq (x-y)(Tx - Ty).$$

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$y = \mathrm{Tan}^{-1} x$　アークタンジェント

$= \arctan x$

$y = \tan x$ の逆関数

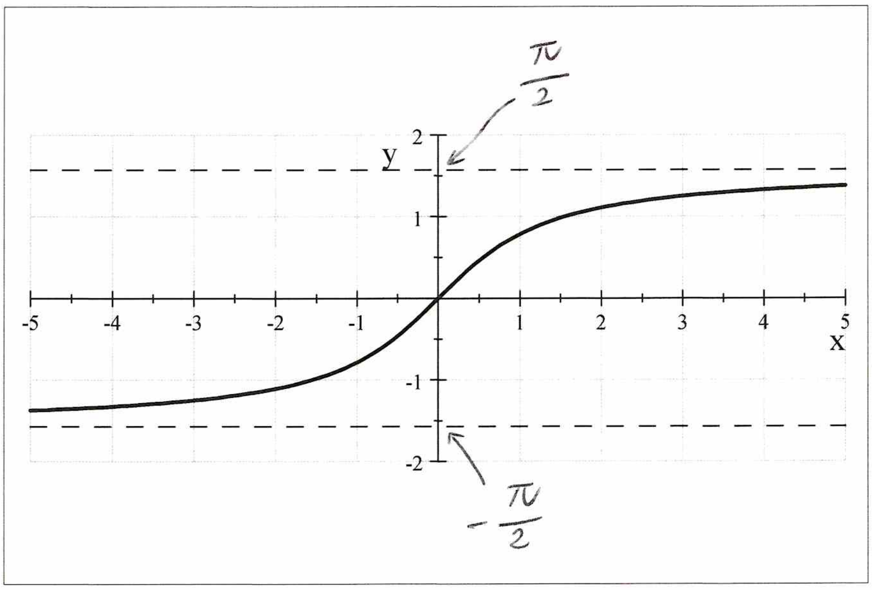
$$y' = \frac{1}{x^2+1}$$

$$|y'| = \left| \frac{1}{x^2+1} \right| \leqq 1$$

$\therefore\ y = \mathrm{Tan}^{-1} x$: nonexpansive (NE)

Furthermore, $y = \mathrm{Tan}^{-1} x$ is monotone increasing.

$y = \arctan x$



Th

H $\mathbb{R}$-Hilbert space

$C \subset H$ $\neq \phi$, closed, convex

$P_C : H \to C$ MP

$\Rightarrow$ ① $\langle x - P_C x, P_C x - y \rangle \geq 0$ $\forall x \in H, y \in C$

② $\|x - P_C x\|^2 + \|P_C x - y\|^2 \leq \|x - y\|^2$ $\forall x \in H, y \in C$

③ P_C : NE, monotone

④ $P_C^2 = P_C$

⑤ $F(P_C) = C$

⑥ $P_C : H \to C$ onto

The nearest point theorem and metric projections

1. 最短距離定理を証明せよ．また，空間の完備性が証明のどこに効いているか確認せよ．

2. ノルム空間などでは，最短距離定理が実ヒルベルト空間での形で成り立つとは限らない．そのことを例をもって示せ．

3. 実ヒルベルト空間Hの非空凸部分集合Cがあるとする．また，$x \in H$, $\bar{x} \in C$とする．このとき，次の3条件が同値であることを証明せよ．

(1) $\|x - \bar{x}\| \leq \|x - y\|$ for all $y \in C$,

(2) $\langle x - \bar{x}, \bar{x} - y \rangle \geq 0$ for all $y \in C$,

(3) $\|x - \bar{x}\|^2 + \|\bar{x} - y\|^2 \leq \|x - y\|^2$ for all $y \in C$.

また，(2)⇒(1)，(3)⇒(1)，(2)⇔(3)は，Cの凸性と$\bar{x} \in C$の仮定がなくても成り立つことを確認せよ．

4. 実ヒルベルト空間$\mathbb{R}$から有界閉区間$[1,2]$への距離射影をグラフを描いて答えよ．

5. 実ヒルベルト空間Hから閉単位球$\overline{U} = \{x \in H \mid \|x\| \leq 1\}$への距離射影を答えよ．

6. Hを実ヒルベルト空間，Cをその空ではない部分集合とする．単調作用素(monotone operator)$T : C \to H$の定義を述べ，$H = \mathbb{R}$の場合について説明せよ．

7. Hを実ヒルベルト空間，Mをその部分空間とする．線型写像$T : C \to H$について，Tが単調作用素であることは，

$$\langle Tx, x \rangle \geq 0 \quad \forall x \in M$$

と同値である．これを示せ．

8. 距離射影は非拡大的で単調であることを示せ．

9. 距離射影の性質を確認せよ．

以下，**おまけ**

10. firmly nonexpansive写像の定義を確認し，それは非拡大的で単調であることを示せ．

11. Cを$\mathbb{R}$の空ではない部分集合で，$T : C \to \mathbb{R}$とする．このとき，Tがfirmly nonexpansive写像であることと，Tが非拡大で単調増加であることは同値である．このことを証明せよ．