Metric spaces

Def

X **MS** (**metric space**)

$\Leftrightarrow d : X \times X \to \mathbb{R}$

(d1) $d(x,y) \geqq 0$; $d(x,y) = 0 \Leftrightarrow x = y$

(d2) $d(x,y) = d(y,x)$

(d3) $d(x,y) \leqq d(x,z) + d(z,y)$

ex

$X = \mathbb{R}$

$d(x,y) = |x-y|$

$\Rightarrow (X,d)$: MS.

(A1) $|x| \geqq 0$; $|x| = 0 \Leftrightarrow x = 0$

(A2) $|\alpha x| = |\alpha||x|$

(A3) $|x+y| \leqq |x| + |y|$

(the fundamental properties of the absolute value $|\cdot|$.)

(X, d) MS

$x_0 \in X$, $r > 0$

- $S_r(x_0) = \{x \in X \mid d(x, x_0) < r\}$

  open sphere with center x_0 and radius $r\ (>0)$.

- $A\ (\subset X)$ open in X.

  $\Leftrightarrow \forall x \in A, \exists r > 0 : S_r(x) \subset A$

  $\Leftrightarrow \forall x \in A, \{x_n\} \subset A^c, x_n \not\to x$

- $B\ (\subset X)$ closed in X.

  $\Leftrightarrow B^c$ is open in X.

  $\Leftrightarrow \forall \{x_n\} \subset B, x_n \to x \in X$

  $\Rightarrow x \in B$.

- $\{x_n\} \subset X$, $x \in X$

  $x_n \to x$

  $\Leftrightarrow d(x_n, x) \to 0$

  $\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : n \geq n_0 \Rightarrow d(x_n, x) < \varepsilon$.

X MS

$A \subset X, \neq \emptyset$

$\bar{A} = \{x \in X \mid \forall \varepsilon > 0,\ S_\varepsilon(x) \cap A \neq \emptyset\}$

<u>closure</u> of A

X MS

$A \subset X, \neq \emptyset$

$\Rightarrow$ <u>Equivalent</u>

① $x \in \bar{A}$

② $\exists \{a_n\} \subset A : a_n \to x$

ex

$X = \mathbb{R}$

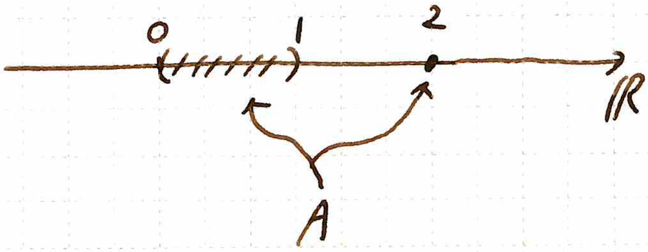
$A = (0,1) \cup \{2\}$

Then, $\bar{A} = [0,1] \cup \{2\}$

For $2 \in \bar{A}$,

$\exists \{a_n\} = \{2, 2, 2, \cdots\} \subset A$:

$a_n \to 2$.



Def.

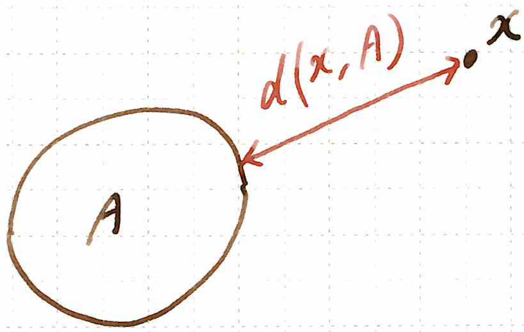
X MS

$A \subset X$, $\neq \emptyset$

$x \in X$

$$d(x, A) \equiv \inf\{d(x,a) \mid a \in A\} \equiv \inf_{a \in A} d(x,a)$$

点と集合との距離



※ $A = \emptyset$, $x \in X$
$\Rightarrow d(x, A) \equiv \infty$

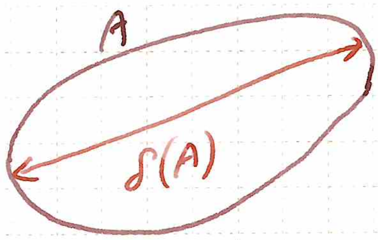
cf.

X MS

$A \subset X, \neq \phi$

$\delta(A) \equiv \sup\{d(x,y) \mid x, y \in A\}$

<u>diameter</u> of A 直径



$$\boxed{\begin{array}{l} A = \phi \\ \Rightarrow \delta(A) \equiv -\infty \end{array}}$$

X MS

$A \subset X$, $\neq \emptyset$

$x \in X$

$\Rightarrow \exists \{a_n\} \subset A : d(x, a_n) \to d(x, A)$

<u>Proof</u>

Define $d \equiv d(x, A)$

$$\equiv \inf\{d(x,a) \mid a \in A\}.$$

Then, $\forall \varepsilon > 0$, $\exists a_\varepsilon \in A$:

$$d \leq d(x, a_\varepsilon) < d + \varepsilon.$$

For $\varepsilon = 1/n > 0$,

$$\exists a_n \in A : d \leq d(x, a_n) < d + \frac{1}{n}$$

Thus, we have

$$\exists \{a_n\} \subset A : d(x, a_n) \to d \equiv d(x, A).$$

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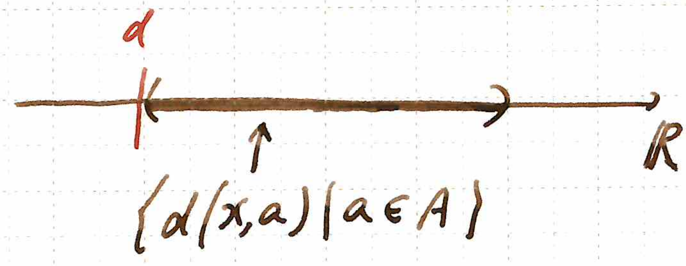


X MS
$A \subset X$ closed in X
$x \in X$
$\Rightarrow$ Equivalent
① $x \in A$
② $d(x, A) = 0$

Proof.

① $\Rightarrow$ ②

From ①,

$$0 \leq d(x, A) \equiv \inf_{a \in A} d(x, a) \overset{①}{\leq} d(x, x) = 0.$$

$\therefore d(x, A) = 0.$ ⌟

② $\Rightarrow$ ①

From ②,

$$\exists \{a_n\} \subset A : d(x, a_n) \to d(x, A) \overset{②}{=} 0.$$

This means that $a_n \to x$.

As $\{a_n\} \subset A$, $a_n \to x \in X$, and A is closed,

we have that $x \in A$. //

Sets of functions

- $X \neq \phi$

  $B(X) = \{ f : X \to \mathbb{R} \mid f \text{ is bdd.} \}$

  $f \in B(X)$

  $\Leftrightarrow \exists L, M \in \mathbb{R} : \forall x \in X, L \leq f(x) \leq M$

- X MS

  $C(X) = \{ f : X \to \mathbb{R} \mid f \text{ is continuous.} \}$

  $f \in C(X)$

  $\Leftrightarrow \forall x \in X, x_n \to x \Rightarrow f(x_n) \to f(x)$

※ $f, g : X \to Y$

$f = g \Leftrightarrow \forall x \in X, f(x) = g(x) \ (\in Y)$

where $X, Y \neq \phi$.

ex.

(i) $X = \mathbb{R}$

Then, $\cdot \sin x, \cos x \in B(X)$

$\cdot x^2, e^x \notin B(X)$

$\cdot \sin x, \cos x, x^2, e^x \in C(X)$

(ii) $X = [0, 1]$

$\sin x, \cos x, x^2, e^x \in B(X) \cap C(X)$

X compact MS
$\Rightarrow C(X) \subset B(X)$

Proof

Let $f \in C(X)$.

As X is compact and f is continuous,

$\exists x^*, x_* \in X : \forall x \in X,$

$$f(x_*) \leq f(x) \leq f(x^*).$$

Therefore, f is bdd on X.

i.e. $f \in B(X)$. //

*: For this result, X can be a top. space.

Th

X compact top. space
$f : X \to \mathbb{R}$ continuous
$\Rightarrow f$ attains maximum and minimum on X.

$X = \mathbb{R}$

- $C^0(\mathbb{R}) = C(\mathbb{R})$
- $C^1(\mathbb{R})$

  $f \in C^1(\mathbb{R})$

  $\Leftrightarrow$ f is continuously differentiable on $\mathbb{R}$

  $\Leftrightarrow$ $\left\{ \begin{array}{l} \cdot\ f \text{ is differentiable} \\ \cdot\ f' \text{ is continuous} \end{array} \right.$

- $C^n(\mathbb{R})$

  $f \in C^n(\mathbb{R})$

  $\Leftrightarrow$ f is n-th order continuously differentiable

  $\Leftrightarrow$ $\left\{ \begin{array}{l} \cdot\ f^{(n-1)} \text{ is differentiable} \\ \cdot\ f^{(n)} \text{ is continuous} \end{array} \right.$

Then,

$$C^0(\mathbb{R}) \supset C^1(\mathbb{R}) \supset C^2(\mathbb{R}) \supset \cdots$$

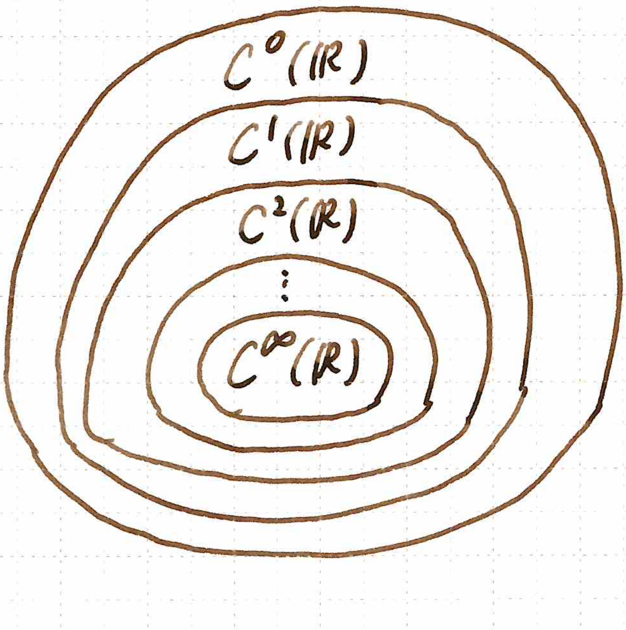
Def.

$$\boxed{C^\infty(\mathbb{R}) \equiv \bigcap_{n=0}^{\infty} C^n(\mathbb{R})}$$

$f \in C^\infty(\mathbb{R})$

$\Leftrightarrow \forall n \in \mathbb{N} \cup \{0\},\ f \in C^n(\mathbb{R})$

infinitely continuously differentiable



$X \neq \phi$

$B(X) = \{ f : X \to \mathbb{R} \mid f \text{ is bdd.} \}$

$d(f,g) = \sup_{x \in X} |f(x) - g(x)| \quad {}^{\forall} f, g \in B(X)$

$\Rightarrow (B(X), d) \quad$ M.S

Proof

(d1) $d(f,g) \geqq 0$; $d(f,g) = 0 \Leftrightarrow f = g$

OK

(d2) $d(f,g) = d(g,f)$ OK

(d3) $d(f,g) \leqq d(f,h) + d(h,g)$

It follows that

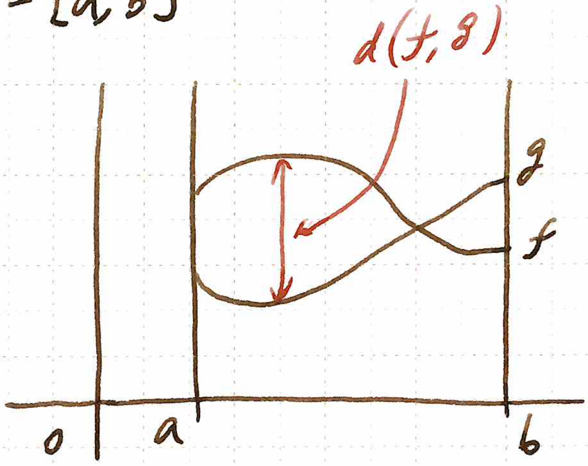
$$
\begin{aligned}
d(f,g) &\equiv \sup_{x \in X} |f(x) - g(x)| \\
&\leqq \sup_{x \in X} \left(|f(x) - h(x)| + |h(x) - g(x)| \right) \\
&\leqq \sup_{x \in X} |f(x) - h(x)| + \sup_{x \in X} |h(x) - g(x)| \\
&= d(f,h) + d(h,g).
\end{aligned}
$$

//

※ We will also show later that $(B(X), d)$ is a complete metric space.

ex

$X = [a, b]$



ex

$X = \mathbb{N}$

Then, $B(\mathbb{N}) = \{\{x_n\} \subset \mathbb{R} \mid \{x_n\} \text{ is bdd}\}$.

Let $\begin{cases} \{a_n\} = \{1, 0, 1, 0, \cdots\} \\ \{b_n\} = \{2, -2, 2, -2, \cdots\} \end{cases}$

Then, $d(\{a_n\}, \{b_n\}) = 2$

ex

$X = \{1, 2\}$

$B(X) = \{f : X \to \mathbb{R} \mid f \text{ is bdd.}\}$

$\therefore B(\{1,2\})$

$= \{f : \{1,2\} \to \mathbb{R} \mid f \text{ is bdd.}\}$

$= \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$

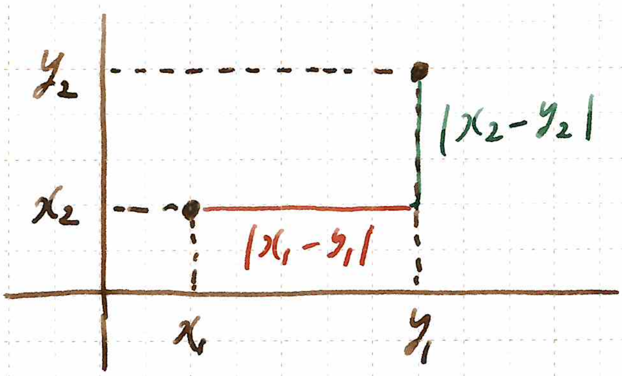
$= \mathbb{R}^2$

In this case,

$d_\infty((x_1, x_2), (y_1, y_2))$

$= \sup_{i \in \{1,2\}} |x_i - y_i|$

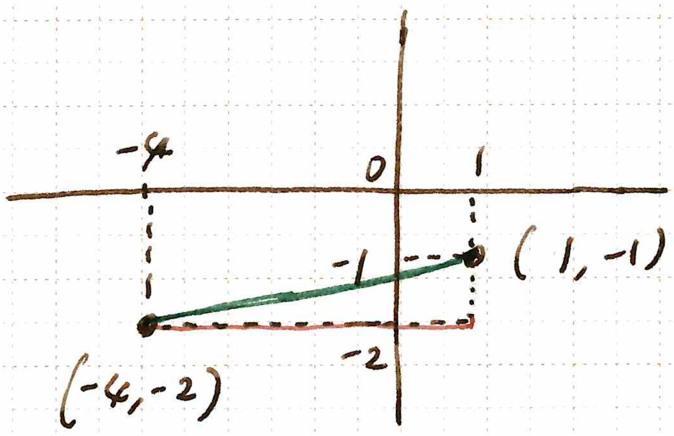
$= \max\{|x_1 - y_1|, |x_2 - y_2|\}$



$d_\infty((1,-1),(-4,-2))$

$= \max\{|1-(-4)|, |-1-(-2)|\}$

$= \max\{5, 1\} = 5$



cf. $d_2((1,-1),(-4,-2))$

$= \sqrt{(1-(-4))^2 + (-1-(-2))^2}$

$= \sqrt{5^2 + 1^2}$

$= \sqrt{26}$

ユークリッドの意味での距離

Def.

X MS

$\{x_n\} \subset X$ <u>Cauchy sequence</u>

$\iff {}^{\forall}\varepsilon > 0, {}^{\exists}n_0 \in \mathbb{N}:$

$m, n \geqq n_0 \Rightarrow d(x_m, x_n) < \varepsilon$

$\iff d(x_m, x_n) \to 0$ as $m, n \to \infty$

X MS

$\{x_n\} \subset X$ convergent

i.e. ${}^{\exists}x \in X : d(x_n, x) \to 0$

$\Rightarrow \{x_n\}$: Cauchy sequence

Def.

X MS

X : complete

$\Longleftrightarrow$ $\{x_n\}$ is a Cauchy sequence.

$\Rightarrow$ $\{x_n\}$ is convergent.

Th (Bolzano-Weierstrass)

$\{x_n\} \subset \mathbb{R}$ bdd

$\Rightarrow {}^{\exists}\{x_{n_i}\} \subset \{x_n\},\ x \in \mathbb{R} : x_{n_i} \to x$

$\Downarrow$

Th

$\mathbb{R}$: complete

Th

$X \neq \phi$

$B(X) = \{f : X \to \mathbb{R} \mid f \text{ is bdd.}\}$

$d(f, g) = \sup_{x \in X} |f(x) - g(x)|$

$\forall f, g \in B(X)$

$\Rightarrow (B(X), d) : \text{CMS}$

Proof (completeness)

Let $\{f_n\} (\subset B(X))$ be a Cauchy sequence.

i.e. $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : \forall m, n \geqq n_0,$

$$\sup_{x \in X} |f_m(x) - f_n(x)| < \varepsilon.$$

i.e. $\forall \varepsilon > 0, \exists n_1 \in \mathbb{N} : \forall m, n \geqq n_1,$

$$\forall x \in X, \ |f_m(x) - f_n(x)| < \varepsilon. \quad -(*)$$

We prove that $\underline{\exists f \in B(X) : d(f_n, f) \to 0.}$

From $(*)$,

$\forall x \in X, \underline{\forall \varepsilon > 0, \exists n_1 \in \mathbb{N} : \forall m, n \geqq n_1,}$

$$\underline{|f_m(x) - f_n(x)| < \varepsilon.}$$

$\therefore \forall x \in X, \underline{\{f_n(x)\} \text{ is a Cauchy sequence in } \mathbb{R}.}$

As $\mathbb{R}$ is complete,

$$\forall x \in X,\ \exists^1 f(x) \in \mathbb{R} : f_n(x) \to f(x). \quad —(**)$$

We have obtained $f : X \to \mathbb{R}$.

We prove that $\left(\begin{array}{l} \text{(i) } f \in B(X) \\ \text{(ii) } d(f_n, f) \to 0. \end{array} \right.$

$$\sup_{x \in X} |f_n(x) - f(x)| \to 0. \quad —(*3)$$

From $(*)$,

$$\forall \varepsilon > 0,\ \exists n_2 \in \mathbb{N} : \forall m, n \geqq n_2,\ \forall x \in X,$$

$$|f_m(x) - f_n(x)| < \frac{\varepsilon}{2} \ (\forall m \geqq n_2).$$

From $(**)$, we obtain as $m \to \infty$,

$$|f(x) - f_n(x)| \leqq \frac{\varepsilon}{2} \quad \forall x \in X$$

$$\therefore \sup_{x \in X} |f(x) - f_n(x)| \leqq \frac{\varepsilon}{2} < \varepsilon.$$

$$\therefore \forall \varepsilon > 0,\ \exists n_2 \in \mathbb{N} : \forall n \geqq n_2,$$

$$\sup_{x \in X} |f_n(x) - f(x)| < \varepsilon. \quad —(*4)$$

$\lrcorner$

$\underline{f \in B(X)}$

i.e. $f : X \to \mathbb{R}$ is bdd.

It follows that

$$\sup_{x \in X} |f(x)|$$

$$\leq \sup_{x \in X} \left(|f(x) - f_{n_2}(x)| + |f_{n_2}(x)| \right)$$

$$\leq \sup_{x \in X} |f(x) - f_{n_2}(x)| + \sup_{x \in X} |f_{n_2}(x)|$$

$$\underset{(*4)}{<} \varepsilon + \underbrace{\sup_{x \in X} |f_{n_2}(x)|}_{(\because)\ f_{n_2} \in B(X)} < \infty .$$

Therefore, $(*3)$ means that

$$d(f_n, f) \to 0.$$

//

Cor

$B([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f \text{ is bdd.}\}$

$d(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)|$ $\quad \forall f, g \in B([a,b])$

$\Rightarrow (B([a,b]), d)$: CMS

Cor

$B(\mathbb{N}) = \{x = \{x_n\} \subset \mathbb{R} \mid \{x_n\} \text{ is bdd.}\}$

$d(x,y) = \sup_{n \in \mathbb{N}} |x_n - y_n|$

where $x = \{x_n\}, y = \{y_n\} \in B(\mathbb{N})$

$\Rightarrow (B(\mathbb{N}), d)$: CMS

Cor

$\mathbb{R}^2$

$d_\infty((x_1, x_2), (y_1, y_2))$

$= \max\{|x_1 - y_1|, |x_2 - y_2|\}$

$\Rightarrow (\mathbb{R}^2, d_\infty)$: CMS

Metric spaces

1. 距離空間の定義と絶対値の基本性質を復習し，実数の集合に差の絶対値として二点間の距離を入れると距離空間になることを証明せよ．

2. 点と集合の距離について説明せよ．また，$A = \{\frac{1}{n} : n \in \mathbb{N}\}$とするとき，次の値を答えよ．

$$(1)\ d(-1,A), \qquad (2)\ d(0,A), \qquad (3)\ d(2,A)$$

3. Aを距離空間Xの空ではない部分集合，xをXの要素とする．このとき，ある点列$\{a_n\}$がAの中に存在し，と$d(x,a_n) \to d(x,A)$なることを示せ．

4. Aを距離空間Xの閉部分集合，xをXの要素とする．このとき，次の2条件は同値である．このことを示せ．

(1) $x \in A$,

(2) $d(x,A) = 0$.

ここで，(1)⇒(2)はAが閉集合でなくてもいえるが，逆はAが閉集合という仮定が必要である．Aが閉集合でないために(2)⇒(1)が成立たなくなる例を挙げよ．

5. 距離空間Xに対して，X上の実数値有界関数の全体$B(X)$と同じく実数値連続関数の全体$C(X)$を考える．

(1) 二次関数$f(x) = x^2$は，$X = \mathbb{R}$のときは$B(X)$に属さないが，$X = (-1,1)$のときは属す．これはどう理解すればよいか．

(2) 次の範囲に属す関数の例を挙げよ．$B(X) \cap C(X)$, $B(X) \cap (C(X))^C$, $(B(X))^C \cap C(X)$, $(B(X) \cup C(X))^C$.

(3) Xをコンパクトな距離空間とすると，$C(X) \subset B(X)$となることを確認せよ．

6. Xを空ではない集合とし，X上の実数値有界関数の全体を$B(X)$と書く．$B(X)$の任意のふたつの要素f,gに対して，

$$d(f,g) = \sup_{x \in X} |f(x) - g(x)|$$

とすると，dはX上の距離になる．このことを証明せよ．また，関数の範囲を有界なものに限定しないと，理論構成上，どのような不都合が生じるか？

※この距離をd_∞と書くこともある．

7. 問題6の記号を踏襲する．

(1) $X = [0,2]$, $f(x) = x$, $g(x) = x^2$とする．このとき，$d(f,g)$の値を答えよ．また，$X = [-2,2]$の場合はどうか．グラフを描いて答えよ．

(2) $X = \mathbb{N}$の場合を考え，$x = \{x_n\} = \{0,1,-1,0,1,-1,\cdots\}$, $y = \{y_n\} = \{-2,0,2,-2,0,2,\cdots\}$とする．このとき，$d(x,y)$の値を答えよ．

(3) $X = \{1,2\}$の場合を考え，$x = (3,1)$, $y = (-1,-5) \in B(X)$とする．このとき，$d(x,y)$の値を答えよ．

8. 完備距離空間の定義を述べ，例を挙げて説明せよ．

9. 問題6において，距離空間$(B(X),d)$が完備になることを証明せよ．