Fixed point approximation

Review

Th

X complete metric space

$T: X \to X$ r-contraction

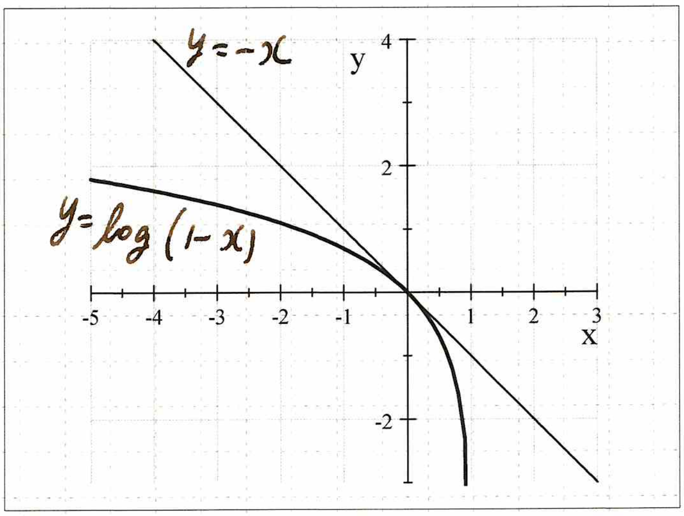
i.e. $\exists r \in (0,1): \forall x, y \in X,$

$$d(Tx, Ty) \leq r\, d(x,y)$$

$\Rightarrow \exists^1 x^* \in F(T):$

$$\forall x \in X,\ T^n x \to x^*$$

$$\log(1-x) \leq -x \quad \forall x < 1$$



$$\{\alpha_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} \alpha_n = \infty$$
$$A > 0 : A\alpha_n < 1$$
$$\Rightarrow \prod_{i=1}^{\infty} (1 - A\alpha_i) = 0$$

Proof.

Define $P_n = \prod_{i=1}^{n} (1 - A\alpha_i) > 0$.

We show that $P_n \to 0$.

It follows that

$$\log P_n = \sum_{i=1}^{n} \log(1 - A\alpha_i)$$
$$\leq -\sum_{i=1}^{n} A\alpha_i = -A \sum_{i=1}^{n} \alpha_i.$$

Thus, $P_n \leq e^{-A \sum_{i=1}^{n} \alpha_i}$.

As $A > 0$ and $\sum_{i=1}^{\infty} \alpha_i = \infty$, we have

$$0 < P_n \leq e^{-A \sum_{i=1}^{n} \alpha_i} \to 0 \ (n \to \infty).$$

$\therefore P_n \to 0$. //

$\xrightarrow{A=1}$

Cor

$$\{\alpha_n\} \subset (-\infty, 1)$$
$$\sum_{n=1}^{\infty} \alpha_n = \infty$$
$$\Rightarrow \prod_{i=1}^{\infty} (1 - \alpha_i) = 0$$

Th

E $\mathbb{R}$-Banach space

$C \subset E \neq \phi$, closed, convex

$T : C \to C$ r-contraction,

where $0 < r < 1$.

$\{\lambda_n\} \subset [0,1]$: $\sum_{n=1}^{\infty} (1-\lambda_n) = \infty$

$$\begin{cases} x_1 \in C \\ x_{n+1} = \lambda_n x_n + (1-\lambda_n) T x_n \quad \forall_{n \in \mathbb{N}} \end{cases}$$

$\Rightarrow x_n \to x^* \in F(T)$

Proof.

As C is complete, $\exists^1 x^* \in F(T)$.

It follows that

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&= \|\lambda_n x_n + (1-\lambda_n) T x_n - x^*\| \\
&= \|\lambda_n (x_n - x^*) + (1-\lambda_n)(T x_n - x^*)\| \\
&\leq \lambda_n \|x_n - x^*\| + (1-\lambda_n) \|T x_n - x^*\| \\
&= \lambda_n \|x_n - x^*\| + (1-\lambda_n) \|T x_n - T x^*\| \\
&\leq \lambda_n \|x_n - x^*\| + (1-\lambda_n) r \|x_n - x^*\|
\end{aligned}$$

$$= \{\lambda_n + (1-\lambda_n)r\} \|x_n - x^*\|$$

$$= \{\lambda_n + r - \lambda_n r\} \|x_n - x^*\|$$

$$= \{r - 1 + 1 + \lambda_n(1-r)\} \|x_n - x^*\|$$

$$= \{1 - (1-r)(1-\lambda_n)\} \|x_n - x^*\|$$

$$\leqq \{1-(1-r)(1-\lambda_n)\} \cdot \{1-(1-r)(1-\lambda_{n-1})\} \cdot \times \|x_{n-1} - x^*\|$$

$$\leqq \cdots$$

$$\leqq \prod_{i=1}^{n} \{1-(1-r)(1-\lambda_i)\} \cdot \|x_1 - x^*\|.$$

As $0 < r < 1$, we have $1 - r > 0$.

Furthermore, $\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$ and

$(1-r)(1-\lambda_n) < 1$.

Thus, $\prod_{i=1}^{\infty}\{1-(1-r)(1-\lambda_i)\} = 0$.

We obtain $x_n \to x^*$.

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※ Mann iteration.

※ If $\lambda_n = 0$ ($\forall n \in \mathbb{N}$), then this coincides with Picard iteration.

Th.

E $\mathbb{R}$-Banach space

$C \subset E \neq \emptyset$, closed, convex

$T: C \to C$ r-contraction,

where $0 < r < 1$.

$\{\mu_n\} \subset [0,1]$

$\{\lambda_n\} \subset [0,1]$: $\sum_{n=1}^{\infty} (1-\lambda_n) = \infty$

$$\begin{cases} x_1 \in C \\ y_n = \mu_n x_n + (1-\mu_n) T x_n \in C \\ x_{n+1} = \lambda_n x_n + (1-\lambda_n) T y_n \in C \quad \forall n \in \mathbb{N} \end{cases}$$

$\Rightarrow x_n \to x^* \in F(T)$

Proof.

As C is complete, $\exists^1 x^* \in F(T)$.

It holds that

$$\begin{aligned} \|y_n - x^*\| &= \|\mu_n x_n + (1-\mu_n) T x_n - x^*\| \\ &\leq \mu_n \|x_n - x^*\| + (1-\mu_n) \|T x_n - x^*\| \\ &= \mu_n \|x_n - x^*\| + (1-\mu_n) \|T x_n - T x^*\| \\ &\leq \mu_n \|x_n - x^*\| + (1-\mu_n) r \|x_n - x^*\| \\ &= \{\mu_n + r(1-\mu_n)\} \|x_n - x^*\|. \end{aligned}$$

Hence,

$$
\begin{aligned}
&\|x_{n+1}-x^*\| \\
&= \|\lambda_n x_n + (1-\lambda_n) T y_n - x^*\| \\
&\leqq \lambda_n \|x_n - x^*\| + (1-\lambda_n)\|T y_n - x^*\| \\
&\leqq \lambda_n \|x_n - x^*\| + (1-\lambda_n) r \|y_n - x^*\| \\
&\leqq \lambda_n \|x_n - x^*\| \\
&\qquad + (1-\lambda_n) r \{\mu_n + r(1-\mu_n)\} \|x_n - x^*\| \\
&= \{\underline{\lambda_n} + (1-\lambda_n) r \mu_n + (1-\lambda_n) r^2 (1-\mu_n)\} \|x_n - x^*\| \\
&= \{\underline{1-(1-\lambda_n)} + (1-\lambda_n) r \mu_n \\
&\qquad\qquad + (1-\lambda_n) r^2 (1-\mu_n)\} \|x_n - x^*\| \\
&= \{1-(1-\lambda_n)[1 - r\mu_n - r^2(1-\mu_n)]\} \|x_n - x^*\| \\
&= \{1-(1-\lambda_n)[1 - r\mu_n - r^2 + r^2\mu_n]\} \|x_n - x^*\| \\
&= \{1-(1-\lambda_n)[1 - r^2 - r\underline{\mu_n}(1-r)]\} \|x_n - x^*\| \quad \mu_n \leqq 1 \\
&\leqq \{1-(1-\lambda_n)[1 - r^2 - r(1-r)]\} \|x_n - x^*\| \\
&= \{1-(1-\lambda_n)(1-r)\} \|x_n - x^*\| \\
&\leqq \cdots \\
&\leqq \prod_{i=1}^{n} \{1-(1-r)(1-\lambda_n)\} \cdot \|x_1 - x^*\| \to 0.
\end{aligned}
$$

$\therefore x_n \to x^*$.

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※ Ishikawa iteration

$\mu_n = 1 \quad \forall n \in \mathbb{N}$

Cor

E $\mathbb{R}$-Banach space

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$\{\lambda_n\} \subset [0,1]$: $\sum_{n=1}^{\infty} (1-\lambda_n) = \infty$

$$\begin{cases} x_1 \in C \\ x_{n+1} = \lambda_n x_n + (1-\lambda_n) T x_n \end{cases}$$

$\Rightarrow x_n \to x^* \in F(T)$

<Mann iteration>

$\lambda_n = 0 \quad \forall n \in \mathbb{N}$

Cor

E $\mathbb{R}$-Banach space

$C \subset E \neq \emptyset$, closed, convex

$T: C \to C$ r-contraction

$\Rightarrow \forall x \in C,\ T^n x \to x^* \in F(T)$

<Picard iteration>

Fixed point approximation

1. $y = \log(1-x)$と$y = -x$のグラフの概形を描き,

$$\log(1-x) \leq -x \quad \forall x < 1$$

となることを確認せよ.

2. $\{\alpha_n\}(\subset \mathbb{R})$を$\sum_{n=1}^{\infty} \alpha_n = \infty$を満たす数列とする. また, $A > 0$は$A\alpha_n < 1$を満たすとする. このとき,

$$\prod_{i=1}^{\infty}(1 - A\alpha_i) = 0$$

となることを示せ.

3. Cを実バナッハ空間Eの空でない閉凸集合とする. また, $T : C \to C$をr-縮小写像とする($0 < r < 1$). 区間$[0,1]$内の数列$\{\lambda_n\}$は$\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$を満たすとする. C内の点列$\{x_n\}$を

$$x_1 \in C : \text{given},$$
$$x_{n+1} = \lambda_n x_n + (1-\lambda_n)Tx_n, \quad \forall n \in \mathbb{N}$$

と定める. このとき, 点列$\{x_n\}$がTの唯一の不動点x^*に収束することを証明せよ. また, Cが閉凸という仮定は, なぜ必要か答えよ.

4. 問題3のどのような特殊ケースを考えると, Picaed近似が得られるか?

おまけ

5. Cを実バナッハ空間Eの空でない閉凸集合とする. また, $T : C \to C$をr-縮小写像とする($0 < r < 1$). 数列$\{\lambda_n\}, \{\mu_n\}$は区間$[0,1]$内の数列とし, $\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$を仮定する. C内の点列$\{x_n\}$を

$$x_1 \in C : \text{given},$$
$$y_n = \mu_n x_n + (1-\mu_n)Tx_n,$$
$$x_{n+1} = \lambda_n x_n + (1-\lambda_n)Ty_n, \quad \forall n \in \mathbb{N}$$

と定める. このとき, 点列$\{x_n\}$がTの唯一の不動点x^*に収束することを証明せよ.