Supremum and infimum

$(X, \leqq)$ ordered set

$\Leftrightarrow$ (O1) $x \leqq x$

(O2) $x \leqq y,\ y \leqq x \Rightarrow x = y$

(O3) $x \leqq y,\ y \leqq z \Rightarrow x \leqq z$

Def.

$\boxed{\begin{array}{l} (X, \leqq) \text{ ordered set} \\ A \subset X,\ \neq \phi \\ \bar{a} \in X \text{ the least upper bound of } A \\ \qquad\quad \text{supremum} \\ \Leftrightarrow \begin{cases} \text{① } {}^{\forall}x \in A,\ x \leqq \bar{a} & \text{(上界)} \\ \text{② } {}^{\forall}x \in A,\ x \leqq a \Rightarrow \bar{a} \leqq a & \text{(最小の)} \end{cases} \end{array}}$

$(X, \leqq)$ ordered set
$A \subset X$, $\neq \phi$
$\bar{a}, \bar{b} = \sup A$
$\Rightarrow \bar{a} = \bar{b}$

uniqueness of sup

Proof.

As $\bar{a} = \sup A$,

$$\begin{cases} ① \ \forall x \in A,\ x \leqq \bar{a}, \\ ② \ \forall x \in A,\ x \leqq a \Rightarrow \bar{a} \leqq a. \end{cases}$$

As $\bar{b} = \sup A$,

$$\begin{cases} (1) \ \forall x \in A,\ x \leqq \bar{b}, \\ (2) \ \forall x \in A,\ x \leqq b \Rightarrow \bar{b} \leqq b. \end{cases}$$

From ① and (2), $\bar{b} \leqq \bar{a}$.

From ② and (1), $\bar{a} \leqq b$.

Using (O2), we obtain $\bar{a} = \bar{b}$. //

Axiom

$A \subset \mathbb{R}$ $\neq \phi$, bdd above
$\Rightarrow \exists \sup A \in \mathbb{R}$

上限公理

Def.

$(X, \leq)$ ordered set

$A \subset X, \neq \emptyset$

$\bar{a} \in X$: <u>maximum</u> of A

$\Leftrightarrow \begin{cases} ① \ \bar{a} \in A \\ ② \ {}^\forall x \in A, \ x \leq \bar{a} \end{cases}$

$A \subset X, \neq \emptyset$

$\bar{a}, \bar{b} \equiv \max A$

$\Rightarrow \bar{a} = \bar{b}$

$\bar{a} = \max A$

$\Rightarrow \bar{a} = \sup A$

$\boxed{\begin{array}{l} (X, \leqq) \text{ ordered set} \\ A \subset X, \neq \phi \\ \bar{a} = \sup A \in A \\ \Rightarrow \bar{a} = \max A \end{array}}$

Proof.

As $\bar{a} = \sup A$,

$$\begin{cases} ① \ {}^{\forall}x \in A,\ x \leqq \bar{a}, \\ ② \ {}^{\forall}x \in A,\ x \leqq a \Rightarrow \bar{a} \leqq a. \end{cases}$$

We show that $\bar{a} = \max A$.

i.e. $\begin{cases} (1) \ \bar{a} \in A \\ (2) \ {}^{\forall}x \in A,\ x \leqq \bar{a} \end{cases}$

OK. //

ex.

$X = \mathbb{R}$

$A = \{1, 2, 3\}$

Then, $3 = \sup A \in A$ and it holds that $3 = \max A$.

$A, B \subset \mathbb{R}, \neq \phi$

$A \subset B$

$\Rightarrow \inf B \overset{①}{\leqq} \inf A \overset{②}{\leqq} \sup A \overset{③}{\leqq} \sup B$

Proof.

Define
$$\begin{cases} \bar{a} \equiv \sup A, \\ \underline{a} \equiv \inf A, \\ \bar{b} \equiv \sup B, \\ \underline{b} \equiv \inf B. \end{cases}$$

We prove ② and ③.

②: Our aim is to show that

$$\underline{a} \leqq \bar{a}.$$

As $A \neq \phi$, choose $x \in A$.

By their definitions,

$\bar{a} \equiv \sup A \Rightarrow x \leqq \bar{a}$,

$\underline{a} \equiv \inf A \Rightarrow \underline{a} \leqq x$.

Therefore, $\underline{a} \leqq x \leqq \bar{a}$.

$\therefore \underline{a} \leqq \bar{a}$.

③: Note that

$$\bar{a} = \sup A$$

$$\Leftrightarrow \begin{cases} (1)\ \forall x \in A,\ x \leqq \bar{a}, \\ (2)\ \forall x \in A,\ x \leqq a \Rightarrow \bar{a} \leqq a. \end{cases}$$

$$\bar{b} = \sup B$$

$$\Leftrightarrow \begin{cases} (i)\ \forall x \in B,\ x \leqq \bar{b} \\ (ii)\ \forall x \in B,\ x \leqq b \Rightarrow \bar{b} \leqq b. \end{cases}$$

We demonstrate that $\underline{\bar{a} \leqq \bar{b}}$.

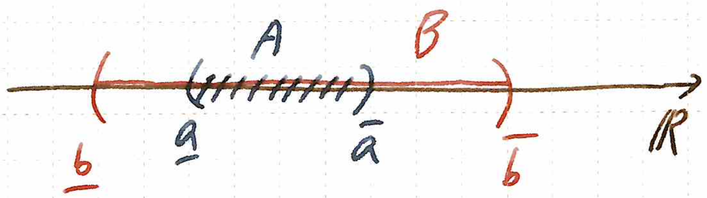
Let $x \in A$.

As $A \subset B$, we have that $x \in (A \subset)\, B$.

From (i), $x \leqq \bar{b}$. $\forall x \in A$.

From (2), $\bar{a} \leqq \bar{b}$.

//



Lemma

$A \subset \mathbb{R}$, $\neq \emptyset$

$\bar{a} = \sup A \in \mathbb{R}$

$\Rightarrow \forall \varepsilon > 0, \exists x \in A : \bar{a} - \varepsilon < x$

Proof:

Suppose by way of contradiction that

$\exists \varepsilon > 0 : \forall x \in A, \ x \leq \bar{a} - \varepsilon.$

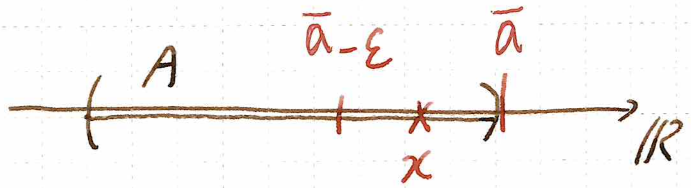
This indicates that

$\bar{a} - \varepsilon$ is an upper bound of A.

As $\bar{a} = \sup A$, we have

$\bar{a} \leq \bar{a} - \varepsilon$ where $\varepsilon > 0$.

This is a contradiction. //



Th

$\{a_n\} \subset \mathbb{R}$ bdd above,
monotone increasing

$\Rightarrow a_n \to \bar{a} \equiv \sup_{k \in \mathbb{N}} a_k$

Proof.

As $\{a_n\} (\subset \mathbb{R})$ is bdd above,
$\exists \bar{a} = \sup_{n \in \mathbb{N}} a_n \in \mathbb{R}$.

$\therefore$ ① $\forall n \in \mathbb{N},\ a_n \leqq \bar{a}$,
② $\forall n \in \mathbb{N},\ a_n \leqq a \Rightarrow \bar{a} \leqq a$.

We prove that
$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : n \geqq n_0 \Rightarrow \bar{a} - \varepsilon < a_n < \bar{a} + \varepsilon$.

Let $\varepsilon > 0$.
From Lemma, $\exists n_0 \in \mathbb{N} : \bar{a} - \varepsilon < a_{n_0}$.
Let $n \geqq n_0$.
As $\{a_n\}$ is monotone increasing, $a_{n_0} \leqq a_n$.
From ①, $\bar{a} - \varepsilon < a_{n_0} \leqq a_n \leqq \bar{a} < \bar{a} + \varepsilon$.

$\therefore \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : n \geqq n_0 \Rightarrow \bar{a} - \varepsilon < a_n < \bar{a} + \varepsilon$.

$\therefore a_n \to \bar{a} \equiv \sup_{k \in \mathbb{N}} a_k$. //

∘ $a_n = \left(1+\frac{1}{n}\right)^n \quad (n \in \mathbb{N})$

$a_1 = \left(1+\frac{1}{1}\right)^1$

$a_2 = \left(1+\frac{1}{2}\right)^2$

$a_3 = \left(1+\frac{1}{3}\right)^3$

$\vdots$

Then, $\{a_n\} (\subset \mathbb{R})$: bdd above,

monotone increasing

$\therefore$ $\{a_n\}$ is convergent.

Def.

$$\boxed{e \equiv \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}$$

ネピアの数

$$\boxed{\begin{array}{l} A \subset \mathbb{R},\ \neq \emptyset \\ \bar{a} = \sup A \in (-\infty, \infty] \\ \Rightarrow \exists \{a_n\} \subset A : a_n \to \bar{a} \end{array}}$$

<u>Proof:</u>

(i) <u>$\bar{a} \in \mathbb{R}$</u>

From <u>Lemma</u>,

$\forall \varepsilon > 0,\ \exists a \in A : \bar{a} - \varepsilon < a.$

Setting $\varepsilon = 1/n > 0$, we have

$\forall n \in \mathbb{N},\ \exists a_n \in A : \bar{a} - \frac{1}{n} < a_n \leq \bar{a}.$

Thus, $a_n \to 0$.

$\therefore \exists \{a_n\} \subset A : a_n \to \bar{a}.$

(ii) <u>$\bar{a} = \infty$</u>

In this case, A is not bdd above.

$\therefore \forall n \in \mathbb{N} : \exists a_n \in A : n < a_n.$

Hence, $a_n \to \infty$.

//

$$\boxed{\begin{array}{l} A \subset \mathbb{R} \neq \emptyset, \text{ bdd above} \\ \alpha \in \mathbb{R} \\ \text{Then,} \\ (1)\ \sup A < \alpha \\ \qquad \Rightarrow \forall x \in A,\ x < \alpha \\ (2)\ \forall x \in A,\ x < \alpha \\ \qquad \Rightarrow \sup A \leq \alpha \end{array}}$$

<u>Proof</u>.

Define $\bar{a} \equiv \sup A \in \mathbb{R}$, that is,

$$\left(\begin{array}{l} ① \ \forall x \in A,\ x \leq \bar{a}, \\ ② \ \forall x \in A,\ x \leq a \Rightarrow \bar{a} \leq a. \end{array} \right.$$

(1): From ①,

$$\forall x \in A,\ x \leq \bar{a} \equiv \sup A < \alpha.$$

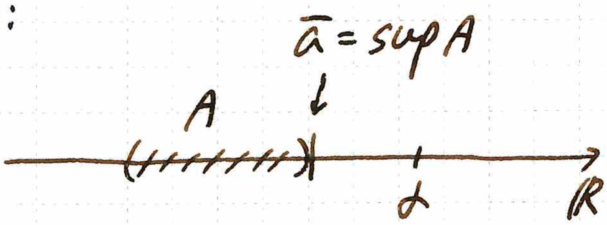
(2): Assume that $\forall x \in A,\ x < \alpha$.

Using ②, we have

$$\bar{a} \equiv \sup A \leq \alpha.$$

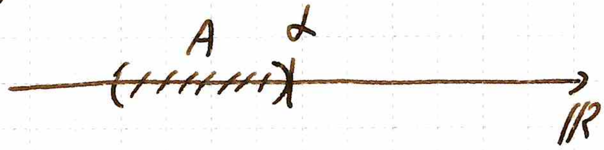
//

(1):



$\forall x \in A,\ x \leqq \bar{a} < \alpha.$

(2):



- $\forall x \in A,\ x < \alpha$
- $\sup A \,(= \alpha) \leqq \alpha.$

$$\boxed{\begin{array}{l}\{a_n\}, \{b_n\} \subset \mathbb{R} \text{ bdd above} \\ \Rightarrow \sup(a_n+b_n) \leqq \sup a_n + \sup b_n\end{array}}$$

<u>Proof.</u>

Define $\begin{cases} \bar{a} \equiv \sup a_n \in \mathbb{R}, \\ \bar{b} \equiv \sup b_n \in \mathbb{R}. \end{cases}$

Then, it holds that

$$\begin{cases} \forall n \in \mathbb{N},\ a_n \leqq \bar{a}, \\ \forall n \in \mathbb{N},\ b_n \leqq \bar{b}. \end{cases}$$

Hence, $a_n + b_n \leqq \bar{a} + \bar{b} \quad \forall n \in \mathbb{N}$.

Therefore, we obtain

$$\begin{aligned} \sup(a_n+b_n) &\leqq \bar{a} + \bar{b} \\ &= \sup a_n + \sup b_n. \end{aligned}$$

//

$$\boxed{\begin{array}{l}\{a_n\}, \{b_n\} \subset \mathbb{R} \text{ bdd below} \\ \Rightarrow \inf a_n + \inf b_n \leq \inf(a_n + b_n)\end{array}}$$

<u>ex</u>

$\{a_n\} = \{1, -1, 1, -1, \cdots\}$

$\{b_n\} = \{-1, 1, -1, 1, \cdots\}$

Then, $a_n + b_n = 0 \quad \forall n \in \mathbb{N}$.

Therefore, it holds that

- $\underbrace{\sup(a_n + b_n)}_{= 0} \lneqq \underbrace{\sup a_n + \sup b_n}_{= 2}$
- $\underbrace{\inf a_n + \inf b_n}_{= -2} \lneqq \underbrace{\inf(a_n + b_n)}_{= 0}$

$$\boxed{\begin{array}{l} \{a_n\} \subset \mathbb{R} \text{ bdd above} \\ \alpha \geqq 0 \\ \Rightarrow \sup \alpha a_n = \alpha \cdot \sup a_n \end{array}}$$

Proof

(i) $\alpha = 0$ OK

(ii) $\alpha > 0$

Define $\bar{a} \equiv \sup a_n \in \mathbb{R}$.

i.e. $\begin{cases} ① \ \forall n \in \mathbb{N},\ a_n \leqq \bar{a} \\ ② \ \forall n \in \mathbb{N},\ a_n \leqq a \Rightarrow \bar{a} \leqq a \end{cases}$

We show that $\underline{\sup \alpha a_n = \alpha \bar{a}}$.

i.e. $\begin{cases} (1) \ \forall n \in \mathbb{N},\ \alpha a_n \leqq \alpha \bar{a} \\ (2) \ \forall n \in \mathbb{N},\ \alpha a_n \leqq a \Rightarrow \alpha \bar{a} \leqq a \end{cases}$

As $\alpha \geqq 0$, (1) follows from ①.

(2): Assume that $\forall n \in \mathbb{N}$, $\alpha a_n \leqq a$.

As $\alpha > 0$, we have $a_n \leqq \dfrac{a}{\alpha}$.

From ②, $\bar{a} \leqq \dfrac{a}{\alpha}$.

$\therefore \alpha \bar{a} \leqq a$.

This completes the proof. //

$$\boxed{\begin{array}{l}\{a_n\} \subset \mathbb{R} \text{ bdd} \\ \alpha \leqq 0 \\ \Rightarrow \sup \alpha a_n = \alpha \cdot \inf a_n\end{array}}$$

Proof

(i) If $\alpha = 0$, the desired result holds.

(ii) $\alpha < 0$

Define $\underline{a} \equiv \inf a_n \in \mathbb{R}$.

i.e. $\begin{cases} ① \ \forall n \in \mathbb{N},\ \underline{a} \leqq a_n \\ ② \ \forall n \in \mathbb{N},\ a \leqq a_n \Rightarrow a \leqq \underline{a} \end{cases}$

We show that $\sup \alpha a_n = \alpha \underline{a}$.

i.e. $\begin{cases} (1) \ \forall n \in \mathbb{N},\ \alpha a_n \leqq \alpha \underline{a} \\ (2) \ \forall n \in \mathbb{N},\ \alpha a_n \leqq a \Rightarrow \alpha \underline{a} \leqq a \end{cases}$

As $\alpha < 0$, (1) follows from ①.

(2): Assume that $\forall n \in \mathbb{N},\ \alpha a_n \leqq a$.

As $\alpha < 0$, $a_n \geqq a/\alpha$.

From ②, $a/\alpha \leqq \underline{a}$.

Thus, $\alpha \underline{a} \leqq a$.

This completes the proof. //

$\{a_n\} \subset \mathbb{R}$ bdd below

$\alpha \geqq 0$

$\Rightarrow \inf \alpha a_n = \alpha \cdot \inf a_n$

$\{a_n\} \subset \mathbb{R}$ bdd

$\alpha \leqq 0$

$\Rightarrow \inf \alpha a_n = \alpha \cdot \sup a_n$

ex

$\{a_n\} = \{1, -1, 1, -1, \cdots\}$

Then, $\{-a_n\} = \{-1, 1, -1, 1 \cdots\}$.

It holds that

- $\sup(-a_n) = -\inf a_n = 1$
- $\inf(-a_n) = -\sup a_n = -1$

Supremum and infimum

1. 順序関係と同値関係の定義を復習せよ.

2. 順序集合の空ではない部分集合について，上限の定義を復習し，その一意性を示せ.

3. 最大元の定義を復習し，その一意性を示せ.

4. 順序集合Xの空ではない部分集合Aの上限$\bar{a}$がAに属しているとき，それはAの最大元になることを示せ.

5. $\mathbb{R}$の空ではないふたつの部分集合A, Bについて，$A \subset B$と仮定すると

$$\inf B \leq \inf A \leq \sup A \leq \sup B$$

となることを証明せよ.

6. $\mathbb{R}$の空ではない部分集合Aの上限を$\bar{a} \in \mathbb{R}$とする．このとき，

$$\forall \varepsilon > 0,\ \exists x \in A : \bar{a} - \varepsilon < x$$

を示せ.

7. $\mathbb{R}$の空ではない部分集合Aの上限を$\bar{a}$とする．このとき，A内に$\bar{a}$に収束してくる数列$\{a_n\}$が存在することを示せ．ただし，Aが上に有界ではない場合は，$a_n \to \infty$となる数列$\{a_n\}$の存在を示せばよい.

8. Aを$\mathbb{R}$の空ではない部分集合とし，上に有界とする．また，αを実数とする．このとき，次の(1) (2)を証明せよ.

(1) $\sup A < \alpha$ならば，任意の$x \in A$に対して$x < \alpha$となる.

(2) 任意の$x \in A$に対して$x < \alpha$ならば，$\sup A \leq \alpha$となる.

9. 上に有界なふたつの数列$\{a_n\}, \{b_n\}$について，

$$\sup_{n\in\mathbb{N}}(a_n + b_n) \leq \sup_{n\in\mathbb{N}} a_n + \sup_{n\in\mathbb{N}} b_n$$

となることを示せ．また，この関係が等号で成り立たない場合の例を挙げよ.

10. 上に有界な数列$\{a_n\}$と$\alpha \geq 0$について，

$$\sup_{n\in\mathbb{N}} \alpha a_n = \alpha \sup_{n\in\mathbb{N}} a_n$$

となることを示せ．また，$\alpha < 0$の場合はこの関係が成り立つとは限らないことを，例を挙げて説明せよ.

11. 有界な数列$\{a_n\}$と$\alpha < 0$について，

$$\sup_{n\in\mathbb{N}} \alpha a_n = \alpha \inf_{n\in\mathbb{N}} a_n$$

となることを示せ.