Inner product, norm, and Hilbert spaces

H pre-Hilbert space over $\mathbb{R}$

$\|x\| = \sqrt{\langle x, x \rangle}$

$\Rightarrow (H, \|\cdot\|)$ normed space

Proof

(N1) $\|x\| \geq 0$; $\|x\| = 0 \Leftrightarrow x = 0$

OK ┘

(N2) $\|\alpha x\| = |\alpha| \, \|x\|$

It follows that

$$\begin{aligned} \|\alpha x\|^2 &= \langle \alpha x, \alpha x \rangle \\ &= \alpha^2 \langle x, x \rangle \\ &= \alpha^2 \|x\|^2. \end{aligned}$$

Thus, we have from (N1) that

$\|\alpha x\| = |\alpha| \, \|x\|$. ┘

$$\boxed{\begin{array}{l} H \text{ pre-Hilbert space over } \mathbb{R} \\ x, y \in H \\ \Rightarrow |\langle x, y\rangle| \leq \|x\| \|y\| \end{array}}$$

Proof

It holds that

$$0 \leq \|tx + y\|^2$$
$$= t^2\|x\|^2 + 2t\langle x, y\rangle + \|y\|^2 \quad \forall t \in \mathbb{R}.$$

Therefore,

$$D/4 = \langle x, y\rangle^2 - \|x\|^2\|y\|^2 \leq 0.$$

$$\therefore \langle x, y\rangle^2 \leq \|x\|^2\|y\|^2.$$

From (N1), $|\langle x, y\rangle| \leq \|x\| \|y\|$.

//

(N3) $\|x + y\| \leq \|x\| + \|y\|$.

It follows that

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$$
$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$
$$= (\|x\| + \|y\|)^2.$$

Using (N1), we obtain the desired result.

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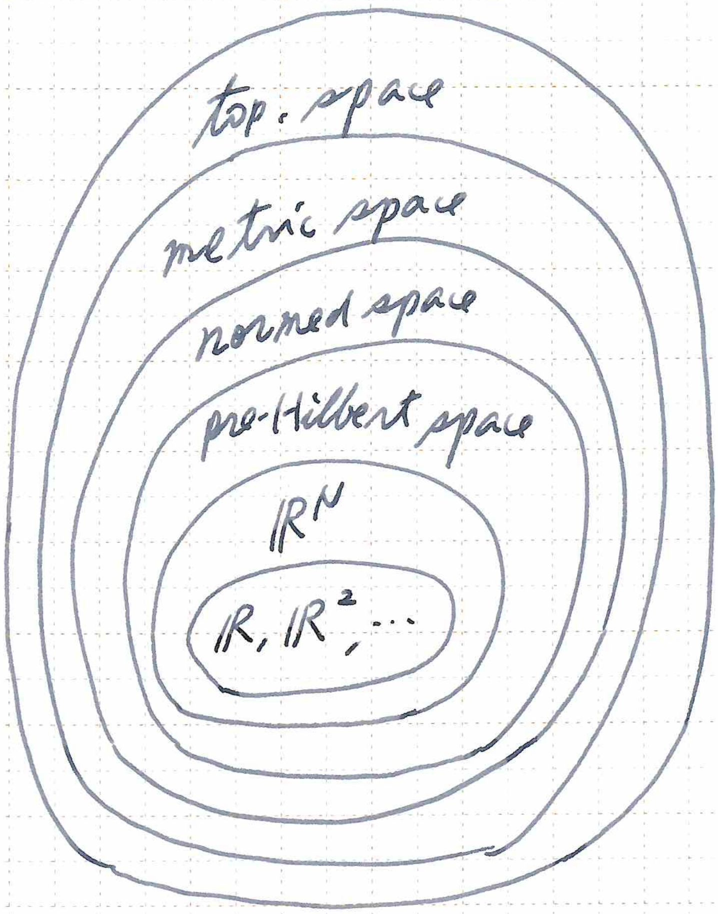

top. space
metric space
normed space
pre-Hilbert space
$\mathbb{R}^N$
$\mathbb{R}, \mathbb{R}^2, \ldots$

Def

H $\mathbb{R}$-Hilbert space

$\Leftrightarrow$ (I) H : $\mathbb{R}$-pre-Hilbert space

(II) H : complete

Euclidean space $\mathbb{R}^N$

$N=3$

$x=(x_1, x_2, x_3)$

$y=(y_1, y_2, y_3)$

(A) sum and scalar multiplication

- $x+y$

$$= (x_1, x_2, x_3) + (y_1, y_2, y_3)$$

$$= (x_1+y_1,\ x_2+y_2,\ x_3+y_3)$$

- $\alpha x = \alpha(x_1, x_2, x_3)$

$$= (\alpha x_1, \alpha x_2, \alpha x_3)$$

(B) inner product

$\langle x, y \rangle$

$$= \langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle$$

$$= x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$= \sum_{n=1}^{3} x_n y_n$$

(C) norm

$$\|x\| = \sqrt{\langle x, x \rangle}$$
$$= \sqrt{\langle (x_1, x_2, x_3), (x_1, x_2, x_3) \rangle}$$
$$= \sqrt{x_1^2 + x_2^2 + x_3^2}$$
$$= \left(\sum_{n=1}^{3} x_n^2 \right)^{\frac{1}{2}}$$

(D) distance (metric)

$$d(x, y) = \|x - y\|$$
$$= \sqrt{\langle x - y, x - y \rangle}$$
$$= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

(E) completeness

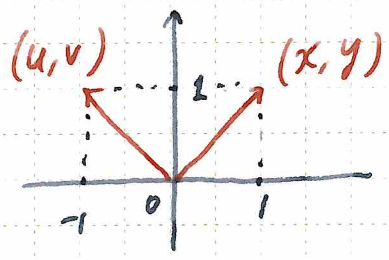
We will prove the completeness
in the next section.

ex

$H = \mathbb{R}^2$

$(x, y) = (1, 1)$

$(u, v) = (-1, 1)$



Then,

$$(x, y) + (u, v) = (0, 2)$$

We verify

$$\boxed{\begin{aligned} &\|(x,y)+(u,v)\|^2 \\ &\quad = \|(x,y)\|^2 + 2\langle (x,y),(u,v)\rangle + \|(u,v)\|^2 \end{aligned}} \quad (*)$$

◦ $\langle (x,y),(u,v) \rangle = xu + yv$

(Euclidean inner product)

In this case,

$$\|(x,y)\| = \sqrt{\langle (x,y),(x,y) \rangle} = \sqrt{x^2+y^2} = \sqrt{2}$$

$$\|(u,v)\| = \sqrt{2}$$

$$\|(x,y)+(u,v)\| = \|(0,2)\| = 2$$

$$\langle (x,y),(u,v) \rangle = 0$$

Therefore, (*) holds.

• $\langle (x,y),(u,v)\rangle = 2xu+yv$

In this case,

$$\|(x,y)\| = \sqrt{\langle (x,y),(x,y)\rangle}$$
$$= \sqrt{2x^2+y^2} = \sqrt{3}$$

$$\|(u,v)\| = \sqrt{3}$$

$$\|(x,y)+(u,v)\| = \|(0,2)\|$$
$$= \sqrt{2\cdot 0^2+2^2} = 2$$

$$\langle (x,y),(u,v)\rangle$$
$$= 2xu+yv = -1.$$

Thus,

$$\|(x,y)+(u,v)\|^2 = 4$$

$$\|(x,y)\|^2+2\langle (x,y),(u,v)\rangle+\|(u,v)\|^2$$
$$= 3-2\cdot 1+3 = 4$$

$\therefore$ (*) holds. //

H (real) pre-Hilbert space

$\{x_n\}, \{y_n\} \subset H$, $x, y \in H$

$x_n \to x$

$y_n \to y$

$\Longrightarrow \langle x_n, y_n \rangle \to \langle x, y \rangle$

Proof

It follows that

$$
\begin{aligned}
&|\langle x_n, y_n\rangle - \langle x, y\rangle| \\
&\leq |\langle x_n, y_n\rangle - \langle x_n, y\rangle| + |\langle x_n, y\rangle - \langle x, y\rangle| \\
&= |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle| \\
&\leq \|x_n\| \|y_n - y\| + \|x_n - x\| \|y\| \\
&\longrightarrow \|x\| \cdot 0 + 0 \cdot \|y\| = 0.
\end{aligned}
$$

//

Cor

H $\mathbb{R}$-pre-Hilbert space

- $f : H \to \mathbb{R}$ defined as follows:

  $$f(x) = \langle x, y \rangle \quad \forall x \in H,$$

  where $y \in H$ is fixed.

- $g : H \to \mathbb{R}$ defined as follows:

  $$g(y) = \langle x, y \rangle \quad \forall y \in H,$$

  where $x \in H$ is fixed.

$\rightarrow$ f, g are linear and continuous.

$\downarrow$

$\ker f$, $\ker g$ $(\subset H)$: closed subspaces

H $\mathbb{R}$-pre-Hilbert space

$x, y \in H$

$\Rightarrow \langle x, y \rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$

Proof

It holds that

$$\begin{aligned} RHS &= \frac{1}{4}\Big[\|x\|^2 + 2\langle x, y\rangle + \|y\|^2 \\ &\qquad - \left(\|x\|^2 - 2\langle x, y\rangle + \|y\|^2\right)\Big] \\ &= \frac{1}{4}\cdot 4\langle x, y\rangle \\ &= \langle x, y\rangle. \end{aligned}$$

//

H $\mathbb{R}$-pre-Hilbert space

$x, y, z, w \in H$

$$\Rightarrow 2\langle x-y, z-w\rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2$$

Proof

It follows that

$$
\begin{aligned}
RHS &= \|x\|^2 - 2\langle x, \underline{\underline{w}}\rangle + \|w\|^2 \\
&\quad + \|y\|^2 - 2\langle y, \underline{z}\rangle + \|z\|^2 \\
&\quad - \left(\|x\|^2 - 2\langle x, \underline{z}\rangle + \|z\|^2\right) \\
&\quad - \left(\|y\|^2 - 2\langle y, \underline{\underline{w}}\rangle + \|w\|^2\right) \\
&= -2\langle x-y, w\rangle - 2\langle y-x, z\rangle \\
&= 2\langle x-y, -w\rangle + 2\langle x-y, z\rangle \\
&= 2\langle x-y, z-w\rangle. \\
&= LHS.
\end{aligned}
$$

//

H $\mathbb{R}$-pre-Hilbert space
$x, y \in H$
$\lambda \in \mathbb{R}$
$\Rightarrow \|\lambda x + (1-\lambda) y\|^2$
$= \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$

Proof

$$\begin{aligned}
RHS &= \lambda\|x\|^2 + (1-\lambda)\|y\|^2 \\
&\quad - \lambda(1-\lambda)\left(\|x\|^2 - 2\langle x, y\rangle + \|y\|^2\right) \\
&= \lambda\left(1-(1-\lambda)\right)\|x\|^2 + (1-\lambda)\left(1-\lambda\right)\|y\|^2 \\
&\quad + 2\lambda(1-\lambda)\langle x, y\rangle \\
&= \lambda^2\|x\|^2 + 2\lambda(1-\lambda)\langle x, y\rangle + (1-\lambda)^2\|y\|^2 \\
&= \|\lambda x + (1-\lambda) y\|^2 = LHS.
\end{aligned}$$

//

Cor

$x, y \in H$
$\lambda \in [0, 1]$
$\Rightarrow \|\lambda x + (1-\lambda) y\|^2$
$\leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2$

$\downarrow \lambda = 1/2$

Cor

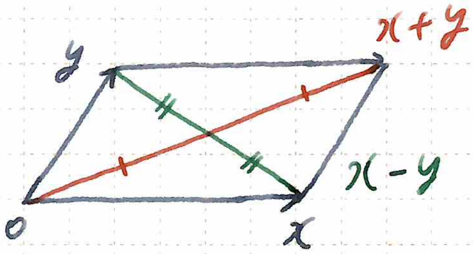
H $\mathbb{R}$-pre-Hilbert space

$x, y \in H$

$\Rightarrow \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

the parallelogram law

平行四辺形公式，中線定理



- A generalization

$x, y, z \in H$

$a, b, c \in \mathbb{R} : a+b+c=1$

$$\Rightarrow \|ax+by+cz\|^2$$
$$= a\|x\|^2 + b\|y\|^2 + c\|z\|^2$$
$$-ab\|x-y\|^2 - bc\|y-z\|^2$$
$$-ca\|z-a\|^2$$

Application

> H $\mathbb{R}$-pre-Hilbert space
> $x, y \in H$
> $\|x\| = \|y\| = \left\| \frac{x+y}{2} \right\|$
> $\Longrightarrow x = y$

Proof

Define $a = \|x\| = \|y\| = \left\| \frac{x+y}{2} \right\|$.

Using the parallelogram law, we have

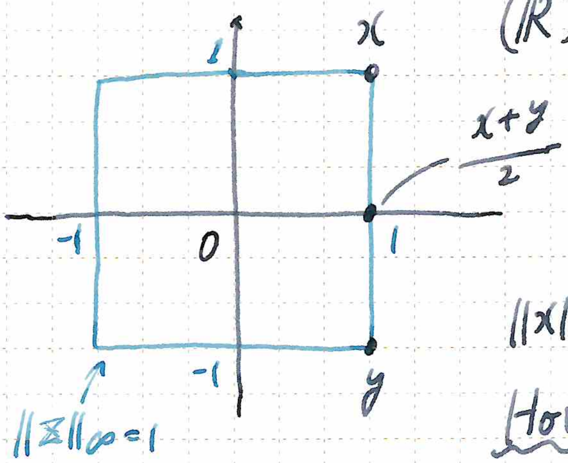
$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

$$\therefore (2a)^2 + \|x-y\|^2 = 2a^2 + 2a^2.$$

$$\therefore \|x-y\|^2 = 0$$

$$\therefore x = y.$$

//



$(\mathbb{R}^2, \|\cdot\|_\infty)$

$\|x\| = \|y\| = \left\| \frac{x+y}{2} \right\|$

However, $x \neq y$

ex

$\mathbb{R}^3$

$x = (1, -1, 2)$

$y = (-2, 1, -1)$

Then,

- $x + y = (-1, 0, 1)$
- $x - y = (3, -2, 3)$
- $\|x\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$
- $\|y\| = \sqrt{(-2)^2 + 1^2 + (-1)^2} = \sqrt{6}$
- $\|x+y\| = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}$
- $\|x-y\| = \sqrt{22}$
- $\langle x, y \rangle = 1 \cdot (-2) + (-1) \cdot 1 + 2 \cdot (-1)$

  $= -5$

Thus,

- $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 = \underline{2}$
- $\|x-y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 = \underline{22}$
- $\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) = \underline{-5}$
- $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 = \underline{24}$

ex

$C([0,1])$

$f(x) = x \quad \forall x \in [0,1]$

$g(x) = x^2 \quad \forall x \in [0,1]$

Then,

$(f+g)(x) = x + x^2$

$(f-g)(x) = x - x^2$

$$\|f\| = \sqrt{\langle f, f \rangle}$$
$$= \sqrt{\int_0^1 x^2 dx} = \frac{1}{\sqrt{3}}$$

$$\|g\| = \sqrt{\langle g, g \rangle}$$
$$= \sqrt{\int_0^1 x^4 dx} = \frac{1}{\sqrt{5}}$$

$$\|f+g\| = \sqrt{\langle f+g, f+g \rangle}$$
$$= \sqrt{\int_0^1 (x+x^2)^2 dx}$$
$$= \sqrt{\int_0^1 (x^4 + 2x^3 + x^2) dx}$$
$$= \sqrt{\left[\frac{x^5}{5} + \frac{x^4}{2} + \frac{x^3}{3}\right]_0^1} = \sqrt{\frac{31}{30}}$$

$$\|f-g\| = \sqrt{\langle f-g, f-g \rangle}$$

$$= \sqrt{\int_0^1 (x-x^2)^2 dx}$$

$$= \sqrt{\int_0^1 (x^4 - 2x^3 + x^2) dx}$$

$$= \sqrt{\left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3}\right]_0^1}$$

$$= \sqrt{\frac{1}{30}}$$

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx = \frac{1}{4}$$

Therefore,

- $\|f+g\|^2 = \|f\|^2 + 2\langle f, g \rangle + \|g\|^2 = \underline{\frac{31}{30}}$
- $\|f-g\|^2 = \|f\|^2 - 2\langle f, g \rangle + \|g\|^2 = \underline{\frac{1}{30}}$
- $\langle f, g \rangle = \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2\right) = \underline{\frac{1}{4}}$
- $\|f+g\|^2 + \|f-g\|^2$

  $= 2\|f\|^2 + 2\|g\|^2 = \underline{\frac{16}{15}}$

Inner product, norm, and Hilbert spaces

1. 実プレ・ヒルベルト空間Hにおいて，$\|x\| = \sqrt{\langle x, x\rangle}$と定義すると$(H, \|\cdot\|)$はノルム空間になる．このことを，シュワルツの不等式とともに証明せよ．

※同様に，複素プレ・ヒルベルト空間も複素ノルム空間になるが，シュワルツの不等式の証明が少々やっかいになるので，以下，本教材では応用上重要な実プレ・ヒルベルト空間を扱う．

2. $\mathbb{R}^2$と$\mathbb{R}^3$について，シュワルツの不等式がどのようなかたちになるか，書き下してみよ．同様に，集合$C([a,b])=\{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous.}\}$において，内積を$\langle f,g\rangle = \int_a^b f(x)g(x)dx$と定義した実プレ・ヒルベルト空間におけるシュワルツの不等式も書いてみよ．

3. $x,y \in H$でもし$y = \alpha x$という形をしていれば，シュワルツの不等式は等号で成り立つ．このことを示せ．ただし，ここで，αはスカラーである．

4. $\{x_n\}, \{y_n\}$を実プレ・ヒルベルト空間H内の点列で，$x_n \to x$，$y_n \to y$とする．このとき，$\langle x_n, y_n\rangle \to \langle x, y\rangle$を示せ．

5. 実プレ・ヒルベルト空間Hの要素yと実数aが与えられたとして，集合

$$C = \{x \in H \mid \langle x, y\rangle \le a\}$$

を定めると，CはHの閉凸集合になる．このことを示せ．

6. 実プレ・ヒルベルト空間Hにおいて，次が成り立つことを証明せよ．

(1) $\langle x,y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$,

(2) $2\langle x-y,\ z-w\rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2$,

(3) $\|ax+by\|^2 = a\|x\|^2 + b\|y\|^2 - ab\|x-y\|^2$，ただし，$a,b$は$a+b=1$を満たす実数である．

7. 問題6の(3)から中線定理を導け．また，(3)を一般化した次の等式を証明せよ．

$$\|ax+by+cz\|^2 = a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x-y\|^2 - bc\|y-z\|^2 - ca\|z-x\|^2,$$

ただし，a,b,cは$a+b+c=1$を満たす実数である．

※ここから，$c=0$と置いた特殊ケースとして問題6の(3)は導かれる．

8. 実プレ・ヒルベルト空間Hの要素x,yについて，$\|x\| = \|x\| = \|\frac{x+y}{2}\|$が成り立つとすると，$x=y$がいえることを示せ．

9. $\mathbb{R}^3$の要素$x=(-2,1,-1)$，$y=(1,-1,-2)$について，次の等式が成り立っていることを確認し，その値を求めよ．

(1) $\|x+y\|^2 = \|x\|^2 + 2\langle x,y\rangle + \|y\|^2$,

(2) $\|x-y\|^2 = \|x\|^2 - 2\langle x,y\rangle + \|y\|^2$,

(3) $\langle x,y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$,

(4) $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

10. 集合$C([0,1])=\{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous.}\}$において，$\langle f,g\rangle = \int_0^1 f(x)g(x)dx$と定義した実プレ・ヒルベルト空間を考える．そのふたつの要素

$$f(x) = x \quad \forall x \in [0,1],$$
$$g(x) = -x \quad \forall x \in [0,1]$$

について，次の等式が成り立っていることを確認し，その値を求めよ．

(1) $\|f+g\|^2 = \|f\|^2 + 2\langle f,g\rangle + \|g\|^2$,

(2) $\|f-g\|^2 = \|f\|^2 - 2\langle f,g\rangle + \|g\|^2$,

(3) $\langle f,g\rangle = \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2\right)$,

(4) $\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$.

解答.

9. (1) 10　(2) 14　(3) -1　(4) 24

10. (1) 0　(2) $\sqrt{4/3}$　(3) $-1/3$　(4) 4/3