Subspaces and linear mappings

Def.

V K-vector space

$A \subset V$, $\neq \emptyset$

A: <u>subspace</u> of V

$\iff \forall x, y \in A$, $\alpha, \beta \in K$,

$\alpha x + \beta y \in A$

Remark

- V, $\{0\}$ $(\subset V)$: subspaces of V.
- A itself becomes a K-vector space.

V K-vector space
$A \subset V$, $\neq \phi$
$\Rightarrow$ Equivalent

① $A (\subset V)$: subspace
i.e. $\forall x, y \in A$; $\alpha, \beta \in K$,
$$\alpha x + \beta y \in A \quad -(*)$$

② $\left\{ \begin{array}{l} \text{(i) } \forall x, y \in A, \ x + y \in A. \\ \text{(ii) } \forall x \in A, \alpha \in K, \ \alpha x \in A \end{array} \right.$

Proof.

① $\Rightarrow$ ②

(i) Letting $\alpha = \beta = 1$ in $(*)$, we have (i).

(ii) Letting $\beta = 0$ in $(*)$, we obtain (ii).

② $\Rightarrow$ ①

Let $x, y \in A$; $\alpha, \beta \in K$.

From (ii), $\alpha x, \beta y \in A$.

From (i), we obtain $\alpha x + \beta y \in A$. //

ex

> $\mathbb{R}^2$
>
> $A = \{(x, y) \in \mathbb{R}^2 \mid x - 2y = 0\}$
>
> $\Rightarrow$ A is a subspace of $\mathbb{R}^2$

Proof

First, note that $A \neq \emptyset$ as $(0,0) \in A$.

Let $(x, y), (u, v) \in A$; $\alpha, \beta \in \mathbb{R}$.

i.e. $\begin{cases} x - 2y = 0 \\ u - 2v = 0 \end{cases}$ — $(*)$

We show that $\alpha(x, y) + \beta(u, v) \in A$.

i.e. $(\alpha x + \beta u, \alpha y + \beta v) \in A$

i.e. $(\alpha x + \beta u) - 2(\alpha y + \beta v) = 0$

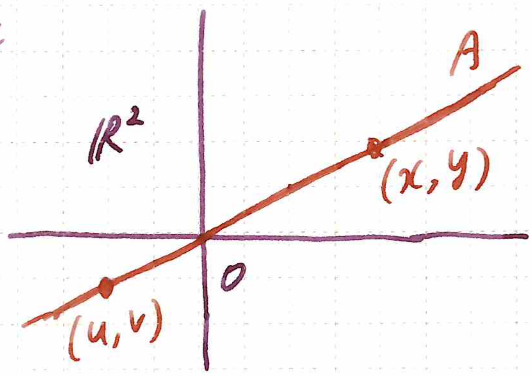
Using $(*)$, we obtain

$(\alpha x + \beta u) - 2(\alpha y + \beta v)$

$= \alpha(x - 2y) + \beta(u - 2v)$

$= 0$.

//



ex

$\mathbb{R}^2$

$B = \{(x, y) \in \mathbb{R}^2 \mid x - 2y + 1 = 0\}$

$\Rightarrow$ B is not a subspace of $\mathbb{R}^2$.

Proof.

It is sufficient to prove that

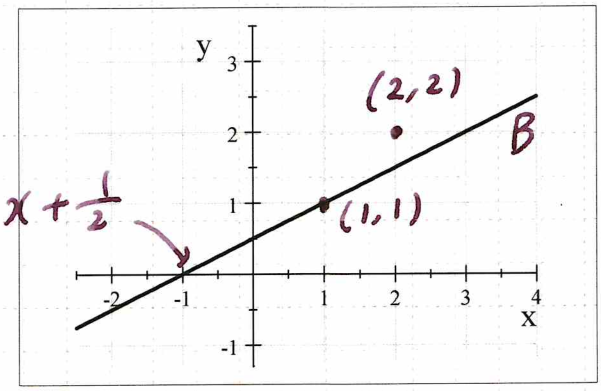
$\exists (x, y) \in B,\ \alpha \in \mathbb{R}$ s.t. $\alpha(x, y) \notin B$.

Let $(x, y) = (1, 1) \in \mathbb{R}^2$ and let $\alpha = 2 \in \mathbb{R}$.

Then, $(1, 1) \in B$.

We have $\alpha(x, y) = (2, 2) \notin B$. //

$y = \frac{1}{2}x + \frac{1}{2}$



ex

$X \neq \emptyset$

$L(X) = \{ f \mid f : X \to \mathbb{R} \}$

$B(X) = \{ f \in L(X) \mid f \text{ is bdd} \}$.

$\Rightarrow B(X)$ is a subspace of $L(X)$.

Proof

Let $f, g \in B(X)$; $\alpha, \beta \in \mathbb{R}$.

i.e. $\begin{cases} \exists M_1 \in \mathbb{R} : \forall x \in X, \ |f(x)| \leq M_1 \\ \exists M_2 \in \mathbb{R} : \forall x \in X, \ |g(x)| \leq M_2 \end{cases}$

We prove that $\alpha f + \beta g \in B(X)$.

i.e. $\exists M \in \mathbb{R} : \forall x \in X,$

$$|\alpha f(x) + \beta g(x)| \leq M.$$

Define $M \equiv |\alpha| M_1 + |\beta| M_2 \in \mathbb{R}$.

Then, it holds true that

$$\begin{aligned} &|\alpha f(x) + \beta g(x)| \\ &\leq |\alpha| \, |f(x)| + |\beta| \, |g(x)| \\ &\leq |\alpha| M_1 + |\beta| M_2 \equiv M. \end{aligned}$$

//

ex

> X compact MS
> $B(X) = \{f \in L(X) \mid f \text{ is bdd.}\}$
> $C(X) = \{f \in L(X) \mid f \text{ is continuous.}\}$
> $\Rightarrow C(X)$ is a subspace of $B(X)$.

Proof.

First, note that as X is compact,

$$C(X) \subset B(X).$$

Choose $f, g \in C(X)$ and $\alpha, \beta \in \mathbb{R}$.

i.e. $\forall x \in X$, $x_n \to x \Rightarrow f(x_n) \to f(x)$ —(*)

$$g(x_n) \to g(x)$$

We prove that $\alpha f + \beta g \in C(X)$.

i.e. $\forall x \in X$, $x_n \to x \Rightarrow (\alpha f + \beta g)(x_n) \to (\alpha f + \beta g)(x)$.

i.e. $\forall x \in X$,

$$x_n \to x \Rightarrow \alpha f(x_n) + \beta g(x_n) \to \alpha f(x) + \beta g(x).$$

From (*), OK. //

ex

$I=(a,b)\ (\subset \mathbb{R})$

$C(I)$

$D(I)=\{f: I\to\mathbb{R} \mid f \text{ is differentiable.}\}$

$\Rightarrow D(I)$ is a subspace of $C(I)$.

Proof.

Let $f, g\in D(I)$; $\alpha, \beta\in\mathbb{R}$.

i.e. $\forall x\in I$, $\left.\begin{array}{l} \exists \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} \\ \exists \lim\limits_{h\to 0} \dfrac{g(x+h)-g(x)}{h} \end{array}\right\} - (*)$

We prove that $\alpha f+\beta g\in D(I)$.

i.e. $\forall x\in I$,

$$\underbrace{\exists \lim_{h\to 0} \frac{(\alpha f+\beta g)(x+h)-(\alpha f+\beta g)(x)}{h}}_{(\star)}$$

We have

$$(\circledast) = \frac{[\alpha f(x+h)+\beta g(x+h)]-[\alpha f(x)+\beta g(x)]}{h}$$

$$= \frac{\alpha[f(x+h)-f(x)]+\beta[g(x+h)-g(x)]}{h}$$

$$= \alpha\cdot\frac{f(x+h)-f(x)}{h}+\beta\cdot\frac{g(x+h)-g(x)}{h}$$

From $(*)$, $(\circledast)$ has the limit as $h\to 0$.

This indicates that

$$\alpha f+\beta g\in D(I).$$

//

Def.

V, W K-vector spaces

$f : V \to W$ <u>linear</u>

$\Leftrightarrow \forall x, y \in V,\ \alpha, \beta \in K,$

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

- $\alpha = \beta = 1$

$$f(x+y) = f(x) + f(y)$$

- $\alpha = 1,\ \beta = -1$

$$f(x-y) = f(x) - f(y)$$

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

$$\Leftrightarrow \begin{cases} \text{(i) } f(x+y) = f(x) + f(y) \\ \text{(ii) } f(\alpha x) = \alpha f(x) \end{cases}$$

$$\boxed{\begin{aligned} &f(\alpha x+\beta y)=\alpha f(x)+\beta f(y) \quad -(*) \\ &\Leftrightarrow f(\alpha x+\beta y+\gamma z) \\ &\qquad =\alpha f(x)+\beta f(y)+\gamma f(z) \quad -(**) \end{aligned}}$$

<u>Proof</u>

$(\Leftarrow)$

Setting $\gamma=0$ in $(**)$, we obtain $(*)$. ⌟

$(\Rightarrow)$

It holds that

$$\begin{aligned} &f(\alpha x+\beta y+\gamma z) \\ &= f((\alpha x+\beta y)+\gamma z) \\ &\overset{(*)}{=} f(\alpha x+\beta y)+\gamma f(z) \\ &\overset{(*)}{=} (\alpha f(x)+\beta f(y))+\gamma f(z) \\ &= \alpha f(x)+\beta f(y)+\gamma f(z). \quad // \end{aligned}$$

ex

$f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = ax \quad \forall x \in \mathbb{R}$$

where $a \in \mathbb{R}$

Then, f is linear.

($\because$)

Let $x, y \in \mathbb{R}$; $\alpha, \beta \in \mathbb{R}$.

We show that

$$\underline{f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).}$$

It follows that

$$
\begin{aligned}
& f(\alpha x + \beta y) \\
& = a(\alpha x + \beta y) \\
& = \alpha \cdot ax + \beta \cdot ay \\
& = \alpha f(x) + \beta f(y).
\end{aligned}
$$

//

ex

$f: \mathbb{R} \to \mathbb{R}$ defined as follows:

$f(x) = x^2 \quad \forall x \in \mathbb{R}$

$\Rightarrow f$ is not linear

$(\because)$

Let $\begin{cases} x = 1 \\ y = 2 \end{cases}$.

Then, $\begin{cases} f(1) = 1 \\ f(2) = 2^2 = 4 \\ f(1+2) = f(3) = 3^2 = 9 \end{cases}$

$\therefore f(1+2) \neq f(1) + f(2)$.

//

ex

$C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous.}\}$

$C^1(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuously differentiable.}\}$

$D : C^1(\mathbb{R}) \to C(\mathbb{R})$ defined by

$Df = f' \quad {}^{\forall} f \in C^1(\mathbb{R})$.

$\Rightarrow D$: linear

Proof

Let $f, g \in C^1(\mathbb{R})$; $\alpha, \beta \in \mathbb{R}$.

i.e $\begin{cases} f, g : \text{differentiable} \\ f', g' : \text{continuous} \end{cases}$

We prove that $D(\alpha f + \beta g) = \alpha Df + \beta Dg$.

Note that $\alpha f + \beta g \in C^1(\mathbb{R})$.

It holds that

$$\begin{aligned} D(\alpha f + \beta g) &= (\alpha f + \beta g)' \\ &= \alpha f' + \beta g' \\ &= \alpha Df + \beta Dg. \end{aligned}$$

//

<u>ex</u>

$f(x) = -2x^3 + 4x^2 - 7x - 2$

Then,

$f'(x) = (-2x^3 + 4x^2 - 7x - 2)'$

$= -2(x^3)' + 4(x^2)'$

$\quad -7(x)' - 2\cdot(1)'$

$= -2\cdot 3x^2 + 4\cdot 2x - 7\cdot 1 - 2\cdot 0$

$= \underline{-6x^2 + 8x - 7}$

ex

$S : C([a,b]) \to \mathbb{R}$ defined by

$$Sf = \int_a^b f(x)\,dx \quad \forall f \in C([a,b]).$$

$\Rightarrow S$: linear

Proof

Let $f, g \in C([a,b])$; $\alpha, \beta \in \mathbb{R}$.

As f, g are continuous,

so is $\alpha f + \beta g$.

Thus, these functions are integrable.

It holds that

$$\begin{aligned} &S(\alpha f + \beta g) \\ &= \int_a^b \big(\alpha f(x) + \beta g(x)\big)\,dx \\ &= \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx \\ &= \alpha \cdot Sf + \beta \cdot Sg. \end{aligned}$$

//

$\boxed{\begin{array}{l} V, W \ K\text{-vector spaces} \\ f : V \to W \text{ linear} \\ \Rightarrow f(0) = 0 \end{array}}$

<u>Proof.</u>

As $V \neq \emptyset$, choose $x \in V$ arbitrarily.
It follows that

$$\begin{aligned} f(0) &= f(x - x) \\ &= f(x) - f(x) \quad \Big\} \ f : \text{linear} \\ &= 0 . \end{aligned}$$

//

※ $f(\underset{\substack{\uparrow \\ V}}{\underset{\sim}{0}}) = \underset{\substack{\uparrow \\ W}}{0}$

Def.

$V \neq \emptyset$

W K-vector space

$f : V \to W$

$\operatorname{Ker} f = \{ x \in V \mid f(x) = 0 \}$

<u>kernel</u> of f 核

<u>ex</u>

$f : \mathbb{R}^2 \to \mathbb{R}^2$

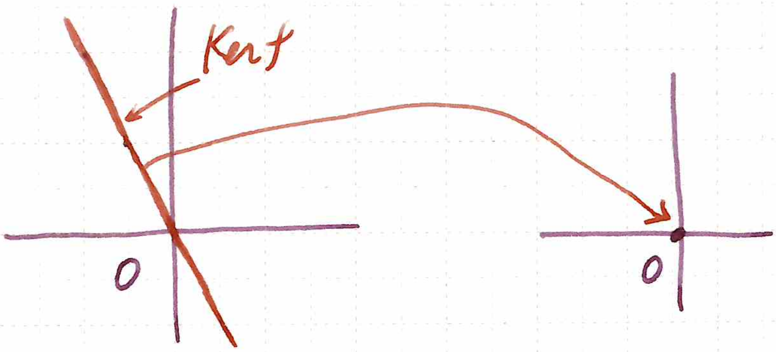
$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad {}^\forall\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

Then, $\mathrm{Ker} f = (?)$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 2x + y \\ -4x - 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow y = -2x$$



Def

$V, W \neq \emptyset$

$f: V \to W$

$$\operatorname{Im} f = f(V)$$
$$= \{ y \in W \mid \exists x \in V : y = f(x) \}$$
$$= \{ f(x) \in W \mid x \in V \}$$

ex

$V = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x \in \mathbb{R} \right\}$

$W = \mathbb{R}^2$

$f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

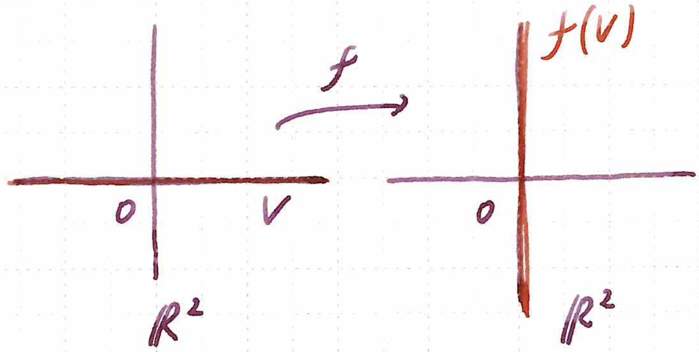
$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

Then, $f(V) = (?)$

Let $\begin{pmatrix} x \\ 0 \end{pmatrix} \in V$.

It holds that

$$f\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ x \end{pmatrix}$$



V, W K-vector spaces
$f: V \to W$ linear
$\Rightarrow \ker f\ (\subset V)$, $\operatorname{Im} f\ (\subset W)$
are subspaces.

Proof

$\ker f\ (\subset V)$: subspace

Let $x, y \in \ker f$; $\alpha, \beta \in K$.

i.e. $f(x) = f(y) = 0\ (\in W)$. — $(*)$

We prove that $\alpha x + \beta y \in \ker f$.

i.e. $f(\alpha x + \beta y) = 0$.

As f is linear, we have

$$
\begin{aligned}
& f(\alpha x + \beta y) \\
&= \alpha f(x) + \beta f(y) && f: \text{linear} \\
&= \alpha \cdot 0 + \beta \cdot 0 && (*) \\
&= 0. \quad \lrcorner
\end{aligned}
$$

$\mathrm{Im} f (\subset W)$: subspace

Let $u, v \in \mathrm{Im} f$; $\alpha, \beta \in K$.

i.e $\begin{cases} \exists x \in V : u = f(x) \\ \exists y \in V : v = f(y) \end{cases}$

We demonstrate that

$\alpha u + \beta v \in \mathrm{Im} f$.

i.e. $\exists z \in V : f(z) = \alpha u + \beta v$.

Define $z = \alpha x + \beta y . \in V$.

Note that as $x, y \in V$ and

V is a vector space,

$z = \alpha x + \beta y \in V$.

It holds that

$$\begin{aligned} \alpha u + \beta v &= \alpha f(x) + \beta f(y) \quad \Big\downarrow f\text{: linear} \\ &= f(\alpha x + \beta y) \\ &= f(z). \end{aligned}$$

//

Def.

V $\mathbb{R}$-vector space

$C \subset V$ convex

$\Longleftrightarrow \forall x, y \in C,\ \lambda \in (0,1),$

$$\lambda x + (1-\lambda) y \in C$$

convex combination

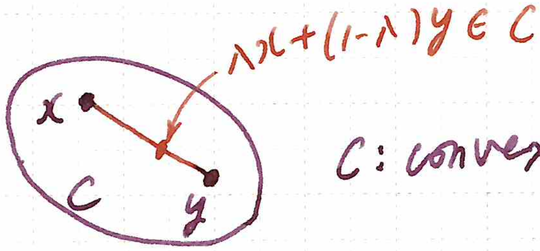
Remarks

- ϕ is convex.
- In $\mathbb{R}$,

  $I (\subset \mathbb{R})$: convex

  $\Longleftrightarrow I$: interval or $I = \{x\}$.
- V ($\mathbb{R}$-vector space) is convex.

ex



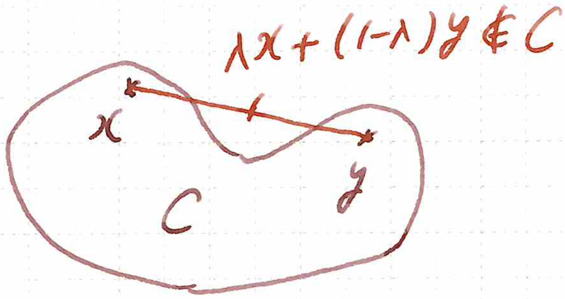
C : convex

$C\,(\subset V)$ is <u>not</u> convex.

$\iff \exists x, y \in C,\ \lambda \in (0,1):$

$\lambda x + (1-\lambda) y \notin C$

<u>ex</u>



ex

$\mathbb{R}^2$

$C = \{(x, y) \in \mathbb{R}^2 \mid x, y \geqq 0,\ x + 2y \leqq 10\}$

$\Rightarrow$ C is convex.

Proof

Let $(x, y), (u, v) \in C$, $\lambda \in (0, 1)$.

i.e. $\begin{cases} x + 2y \leqq 10, & x, y \geqq 0 \\ u + 2v \leqq 10, & u, v \geqq 0 \end{cases}$

We prove that $\lambda(x, y) + (1-\lambda)(u, v) \in C$.

i.e. $(\lambda x + (1-\lambda)u,\ \lambda y + (1-\lambda)v) \in C$

i.e. $\begin{cases} \lambda x + (1-\lambda)u \geqq 0 \\ \lambda y + (1-\lambda)v \geqq 0 \\ (\lambda x + (1-\lambda)u) + 2(\lambda y + (1-\lambda)v) \leqq 10. \end{cases}$

As $x, y, u, v \geqq 0$ and $\lambda \in (0, 1)$,

it holds that

$\begin{cases} \lambda x + (1-\lambda)u \geqq 0 \\ \lambda y + (1-\lambda)v \geqq 0. \end{cases}$

Furthermore,

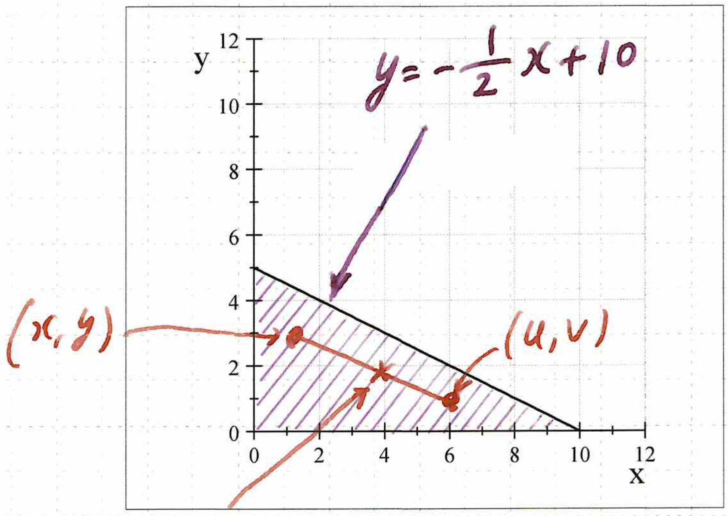
$$(\lambda x+(1-\lambda)u)+2(\lambda y+(1-\lambda)v)$$

$$=\lambda\underbrace{(x+2y)}_{\leq 10}+(1-\lambda)\underbrace{(u+2v)}_{\leq 10}$$

$$\leq \lambda\cdot 10+(1-\lambda)\cdot 10$$

$$=10.$$

//



$$\lambda(x,y)+(1-\lambda)(u,v)$$

$$=(\lambda x+(1-\lambda)u,\ \lambda y+(1-\lambda)v)$$

V $\mathbb{R}$-vector space
$C \subset V$ convex
$x, y, z \in C$
$a, b, c \in (0,1) : a+b+c=1$
$\Rightarrow ax+by+cz \in C$

Proof

It follows that

$$ax+by+cz = (a+b)\left(\frac{a}{a+b}x + \frac{b}{a+b}y\right) + cz.$$

As $\frac{a}{a+b}, \frac{b}{a+b} \in (0,1)$, it holds that

$$\frac{a}{a+b}x + \frac{b}{a+b}y \in C.$$

As $z \in C$ and $(a+b)+c=1$, we obtain

$$ax+by+cz = (a+b)\underbrace{\left(\frac{a}{a+b}x + \frac{b}{a+b}y\right)}_{\in C} + c\underset{\in C}{\underline{\underline{z}}}$$

$$\in C.$$

//

Subspaces and linear mappings

1. 実ベクトル空間$\mathbb{R}^2$において，$M = \{(x,y) \in \mathbb{R}^2 \mid ax + by = 0\}$（ただし，$a, b \in \mathbb{R}$）は$\mathbb{R}^2$の部分空間である．このことを示せ．

2. Xを空ではない集合とし，X上の実数値関数の全体を$L(X)$，実数値有界関数の全体を$B(X)$と書く．すなわち，

$$L(X) = \{f \mid f : X \to \mathbb{R}\},$$
$$B(X) = \{f \in L(X) \mid f \text{ is bounded.}\}$$

である．$B(X)$が$L(X)$の部分空間になることを証明せよ．

3. X, Yをベクトル空間Vの部分空間とする．
(1) 共通部分$X \cap Y$もVの部分空間になることを示せ．
(2) 合併集合$X \cup Y$はVの部分空間になるとは限らない．このことを例を挙げて説明せよ．

4. Xをコンパクトな距離空間とし，$L(X)$と$B(X)$を問題2で定めた集合とする．また，

$$C(X) = \{f \in L(X) \mid f \text{ is continuous.}\}$$

とする．このとき，$C(X)$は$B(X)$の部分空間になることを示せ．

5. 線型写像について，例を挙げて説明せよ．

6. 線形写像fについて，次の同値性を証明せよ．

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$
$$\Leftrightarrow f(\alpha x + \beta y + \gamma z) = \alpha f(x) + \beta f(y) + \gamma f(z)$$

7. 線形写像fについて，$f(0) = 0$を証明せよ．

8. 写像fについて，核（kernel）$\ker f$と像（image）$\mathrm{Im} f$の定義を述べよ．また，fが線形写像なら，$\ker f$と$\mathrm{Im} f$はそれぞれ定義域と値域の部分空間になることを証明せよ．

9. 正の定数a, b, c, dが与えられているとすると，実ベクトル空間$\mathbb{R}^3$の部分集合

$$C = \{(x,y,z) \in \mathbb{R}^3 \mid ax + by + cz \leq d,\ x,y,z \geq 0\}$$

は凸集合である．このことを示せ．

10. Cを実ベクトル空間Vの凸部分集合でx, y, zをCの要素とする．また，$a, b, c \in (0, 1)$は$a + b + c = 1$を満たす実数とする．このとき，$ax + by + cz \in C$となることを示せ．

11. 実ベクトル空間$L(\mathbb{R}) = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$において，部分集合

$$C = \{f \in L(\mathbb{R}) \mid f \text{ is monotone increasing.}\}$$

は凸集合である．このことを示せ．ただし，fが単調増加（monotone increasing）とは，$x \leq y \Rightarrow f(x) \leq f(y)$が満足されることである．