Normed spaces

Def

E <u>K-normed space</u>

$(K = \mathbb{C} \text{ or } \mathbb{R})$

$\Leftrightarrow$ (I) E : K-vector space

(II) $\|\cdot\| : E \to \mathbb{R}$ s.t.

(N1) $\|x\| \geqq 0$; $\|x\| = 0 \Leftrightarrow x = 0$

(N2) $\|\alpha x\| = |\alpha| \, \|x\|$

(N3) $\|x + y\| \leqq \|x\| + \|y\|$

※

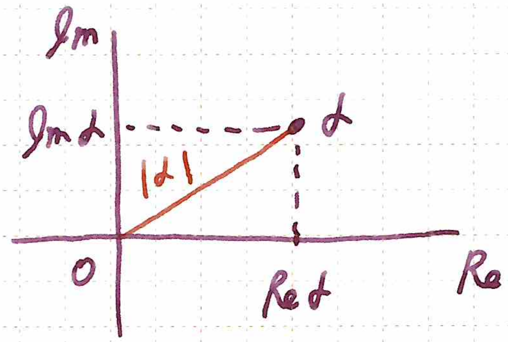
$$\text{(N1)}\ \|x\| \underset{\uparrow \atop \mathbb{R}}{\geqq 0}\,;\ \ \|x\| \underset{\uparrow \atop \mathbb{R}}{= 0} \iff x \underset{\uparrow \atop E}{= 0}$$

the equal sign on E

the equal sign on $\mathbb{R}$

※ If $K=\mathbb{C}$, then $\alpha \in \mathbb{C}$.

$$|\alpha| = \sqrt{(\mathrm{Re}\,\alpha)^2 + (\mathrm{Im}\,\alpha)^2}$$



__ex__

$\mathbb{R}$ with $|\cdot|$

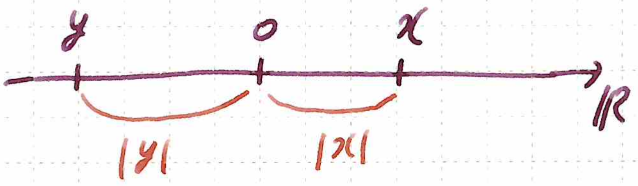
(the absolute value)

the fundamental properties of $|\cdot|$

(A1) $|x| \geqq 0$; $|x| = 0 \iff x = 0$

(A2) $|\alpha x| = |\alpha| |x|$

(A3) $|x + y| \leqq |x| + |y|$



原点0からの距離

ex

$\mathbb{R}^2$

$x = (a, b) \in \mathbb{R}^2$

$\|x\|_2 = \sqrt{a^2 + b^2}$

$\Rightarrow (\mathbb{R}^2, \|\cdot\|_2)$ $\mathbb{R}$-normed space

Proof

(N1) $\|x\|_2 \geqq 0$; $\|x\|_2 = 0 \Leftrightarrow x = 0$.

OK.

(N2) $\|\alpha x\|_2 = |\alpha| \|x\|_2$, where $\alpha \in \mathbb{R}$.

It follows that

$$\|\alpha x\|_2 = \|(\alpha a, \alpha b)\|^2$$
$$= \sqrt{(\alpha a)^2 + (\alpha b)^2}$$
$$= \sqrt{\alpha^2(a^2 + b^2)}$$
$$= |\alpha| \sqrt{a^2 + b^2} = |\alpha| \|x\|_2.$$

(N3) $\|x + y\|_2 \leqq \|x\|_2 + \|y\|_2$

We will show this in the next chapter using Schwarz's inequality.

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Th

E K-normed space
$d(x,y) \equiv \|x-y\| \quad \forall x,y \in E$
$\Rightarrow (E,d)$ MS

Proof

(d1) $d(x,y) \geq 0$; $d(x,y)=0 \Leftrightarrow x=y$.

OK

(d2) $d(x,y)=d(y,x)$

It follows that

$$\begin{aligned} d(x,y) &= \|x-y\| \\ &= \|-(y-x)\| \\ &= |-1|\,\|y-x\| \quad \downarrow (N2) \\ &= \|y-x\| = d(y,x). \end{aligned}$$

(d3) $d(x,y) \leq d(x,z)+d(z,y)$

We have

$$\begin{aligned} d(x,y) &= \|x-y\| \\ &\leq \|x-z\|+\|z-y\| \\ &= d(x,z)+d(z,y). \end{aligned}$$

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※ If $K=\mathbb{C}$, then $-1=-1+0i$.

$$\boxed{x - y = -(y-x)}$$

$(\because)$

It is sufficient to prove that

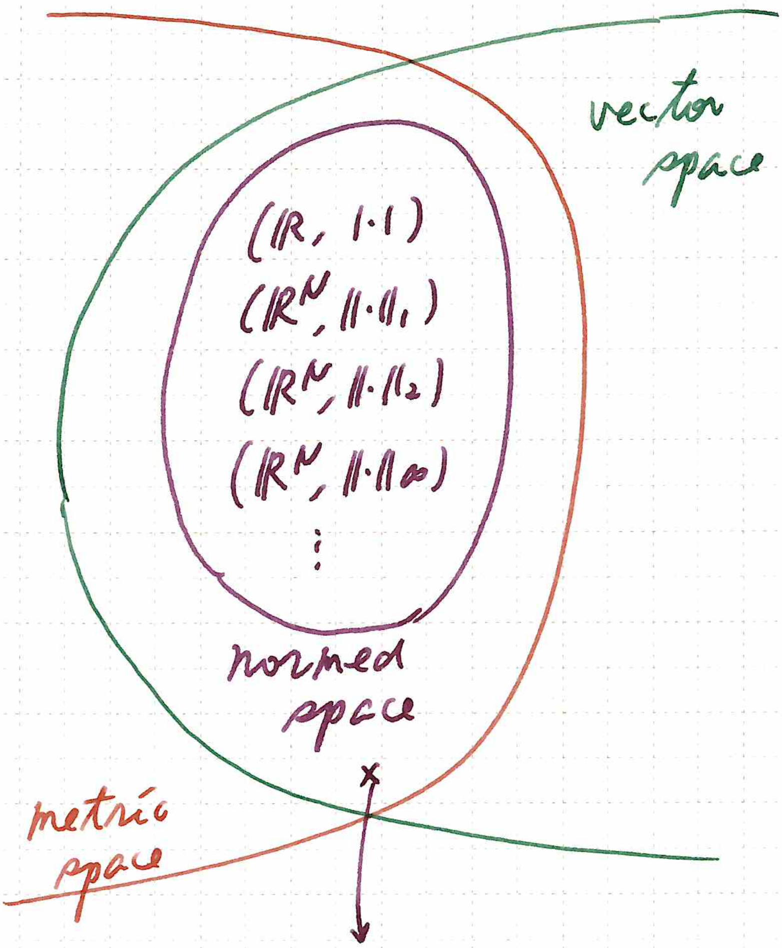
$(y-x)+(x-y)=0$.

It holds that

$$
\begin{aligned}
&(y-x)+(x-y) \\
&= y+(-x+x)+(-y) \\
&=(y+0)+(-y) \\
&= y+(-y) \\
&= 0. \quad //
\end{aligned}
$$

※ $\|x\| = d(x,0)$

zero からの距離


vector space
metric space
normed space
$(\mathbb{R}, |\cdot|)$
$(\mathbb{R}^N, \|\cdot\|_1)$
$(\mathbb{R}^N, \|\cdot\|_2)$
$(\mathbb{R}^N, \|\cdot\|_\infty)$
$\vdots$

next page

V vector space

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

discrete metric

$\Rightarrow (V, d)$ is a metric space

Def

$$\boxed{\|x\| = d(x, 0)}$$

Then, $\|\cdot\|$ is not a norm on V.

Let $\begin{cases} x \in V ; \ x \neq 0, \\ \alpha = \frac{1}{2} \in K. \end{cases}$

It follows that

- $\|\alpha x\| = \left\|\frac{1}{2}x\right\|$
  $= d\left(\frac{1}{2}x, 0\right) = 1$
- $|\alpha| \|x\| = \frac{1}{2} \cdot d(x, 0) = \frac{1}{2}.$

$\therefore \|\alpha x\| \neq |\alpha| \|x\|.$

//

ex

$\mathbb{R}^2$

$x = (a, b) \in \mathbb{R}^2$

$\|x\|_1 = |a| + |b|$

$\Rightarrow (\mathbb{R}^2, \|\cdot\|_1)$ $\mathbb{R}$-normed space

Proof

(N1) $\|x\|_1 \geqq 0$; $\|x\|_1 = 0 \Leftrightarrow x = 0$

OK

(N2) $\|\alpha x\|_1 = |\alpha| \|x\|_1$

It holds true that

$$
\begin{aligned}
\|\alpha x\|_1 &= \|\alpha(a, b)\|_1 \\
&= \|(\alpha a, \alpha b)\|_1 \\
&= |\alpha a| + |\alpha b| \\
&= |\alpha||a| + |\alpha||b| \\
&= |\alpha|(|a| + |b|) \\
&= |\alpha| \|(a, b)\|_1 \\
&= |\alpha| \|x\|_1 .
\end{aligned}
$$

$\lrcorner$

(N3) $\|x+y\|_1 \leq \|x\|_1 + \|y\|_1$

Let $\begin{cases} x=(a,b) \in \mathbb{R}^2, \\ y=(c,d) \in \mathbb{R}^2. \end{cases}$

It follows that

$$\begin{aligned}
&\|x+y\|_1 \\
&= \|(a,b)+(c,d)\|_1 \\
&= \|(a+c,\ b+d)\|_1 \\
&= |a+c| + |b+d| \\
&\leq |a| + |c| + |b| + |d| \\
&= |a| + |b| + |c| + |d| \\
&= \|(a,b)\|_1 + \|(c,d)\|_1 \\
&= \|x\|_1 + \|y\|_1.
\end{aligned}$$

//

ex

$X \neq \emptyset$

$B(X) = \{ f : X \to \mathbb{R} \mid f \text{ is bdd.} \}$

$\|f\|_\infty = \sup_{x \in X} |f(x)| \quad {}^\forall f \in B(X).$

$\Rightarrow (B(X), \|\cdot\|_\infty)$ $\mathbb{R}$-normed space

Proof

(N1) $\|f\|_\infty \geqq 0 ; \|f\|_\infty = 0 \Leftrightarrow f = 0$

OK.

(N2) $\|\alpha f\|_\infty = |\alpha| \|f\|_\infty$

It follows that

$$\begin{aligned} \|\alpha f\|_\infty &= \sup_{x \in X} |\alpha f(x)| \\ &= \sup_{x \in X} |\alpha| |f(x)| \\ &= |\alpha| \cdot \sup_{x \in X} |f(x)| \\ &= |\alpha| \|f\|_\infty . \end{aligned}$$

┘

(N3) $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

We have the following:

$$\|f+g\|_\infty = \sup_{x \in X} |(f+g)(x)|$$
$$= \sup_{x \in X} |f(x) + g(x)|$$
$$\leq \sup_{x \in X} \left(|f(x)| + |g(x)| \right)$$
$$\leq \sup_{x \in X} |f(x)| + \sup_{x \in X} |g(x)|$$
$$= \|f\|_\infty + \|g\|_\infty$$

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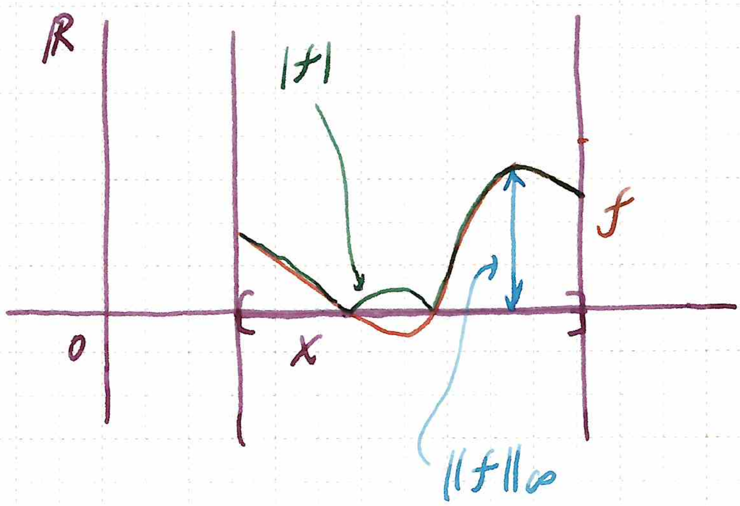


Cor

$B([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f \text{ is bdd.}\}$

$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)| \quad \forall f \in B([a,b])$

$\Rightarrow (B(X), \|\cdot\|_\infty)$ $\mathbb{R}$-normed space

Cor

$B(\mathbb{N}) = \{x = \{x_n\} \subset \mathbb{R} \mid \{x_n\} \text{ is bdd.}\}$

$\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n| \quad \forall x = \{x_n\} \in B(\mathbb{N})$

$\Rightarrow (B(\mathbb{N}), \|\cdot\|_\infty)$ $\mathbb{R}$-normed space

Cor

$B(\{1,2\}) = \mathbb{R}^2$

$\|(x_1, x_2)\|_\infty = \max\{|x_1|, |x_2|\}$

where $x = (x_1, x_2) \in \mathbb{R}^2$

$\Rightarrow (B(\{1,2\}), \|\cdot\|_\infty)$

$\mathbb{R}$-normed space

ex

$\mathbb{R}^2$

$x = (-1, 2)$

$y = (-2, -1)$

Then, $d_1(x, y) = \|x-y\|_1 = (?)$

$d_2(x, y) = \|x-y\|_2 = (?)$

$d_\infty(x, y) = \|x-y\|_\infty = (?)$

It holds that $x-y = (1, 3)$

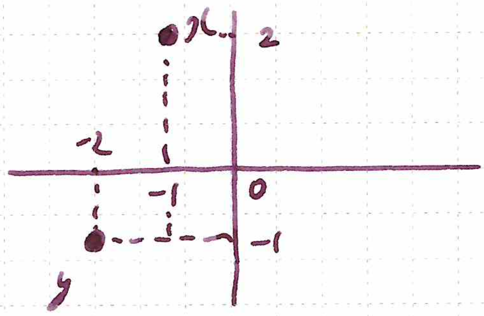
Therefore,

$d_1(x, y) = \|x-y\|_1 = |1| + |3| = 4$

$d_2(x, y) = \|x-y\|_2 = \sqrt{1^2+3^2} = \sqrt{10}$

$d_\infty(x, y) = \|x-y\|_\infty$

$= \max\{|1|, |3|\} = 3$

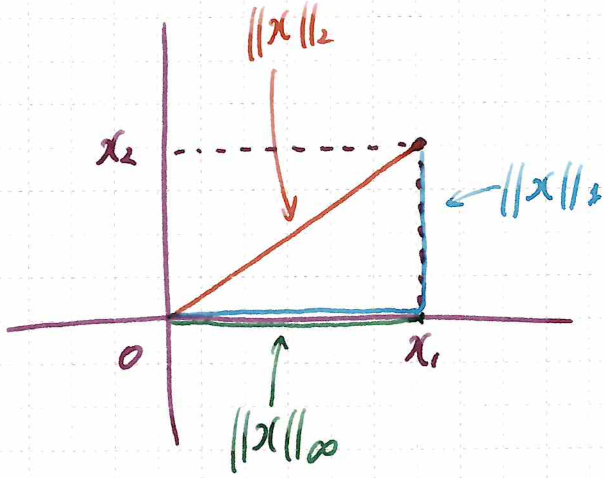


norms on $\mathbb{R}^2$

$$\|x\|_1 = |x_1| + |x_2|$$

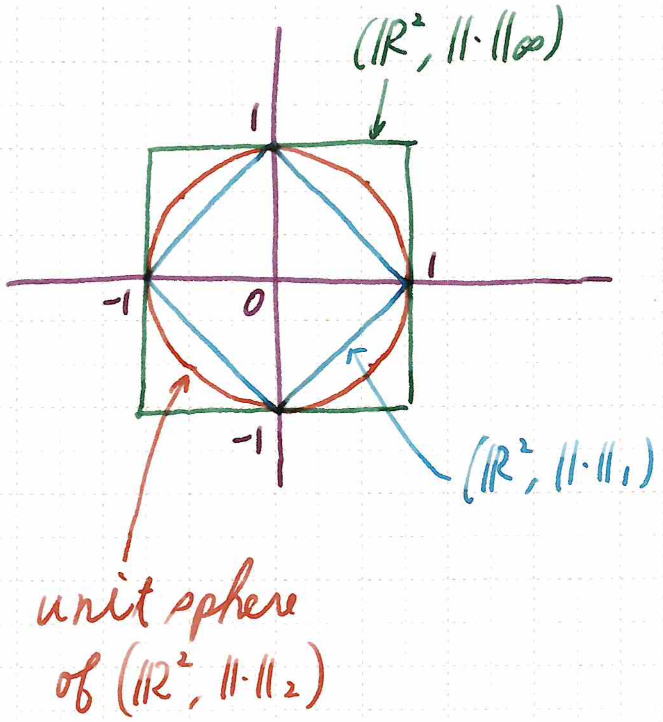
$$\|x\|_2 = \sqrt{x_1^2 + x_2^2}$$

$$\|x\|_\infty = \max\{|x_1|, |x_2|\}$$



unit sphere

$$U \equiv \{ x \in E \mid \|x\| = 1 \}$$



$(E, \|\cdot\|)$ normed space

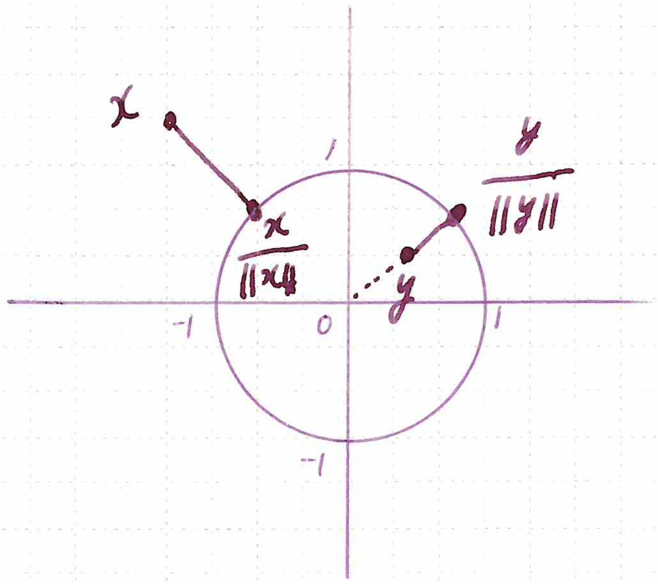
$x \in E,\ x \neq 0$

$\Rightarrow \dfrac{x}{\|x\|} \in U$

i.e. $\left\| \dfrac{x}{\|x\|} \right\| = 1$

ex

$E = \mathbb{R}^2$



E $\mathbb{R}$-normed space
$S \equiv S_1(0) \equiv \{x \in E \mid \|x\| < 1\}$
$\Longrightarrow S$ is convex.

Proof.

Let $x, y \in S$; $\lambda \in (0,1)$.

i.e. $\|x\| < 1$, $\|y\| < 1$ — $(*)$

We prove that $\lambda x + (1-\lambda) y \in S$.

i.e. $\|\lambda x + (1-\lambda) y\| < 1$

It holds that

$$
\begin{aligned}
&\|\lambda x + (1-\lambda) y\| \\
&\le \|\lambda x\| + \|(1-\lambda) y\| && \downarrow (N3) \\
&= |\lambda| \|x\| + |1-\lambda| \|y\| && \downarrow (N2) \\
&= \lambda \|x\| + (1-\lambda) \|y\| && \downarrow \lambda \in (0,1) \\
&< \lambda \cdot 1 + (1-\lambda) \cdot 1 && \downarrow (*) \\
&= 1.
\end{aligned}
$$

//

E $\mathbb{K}$-normed space
$x, y \in E$
$\Rightarrow \big| \|x\| - \|y\| \big| \leqq \|x - y\|$

Proof

It is sufficient to show that
$-\|x-y\| \leqq \|x\| - \|y\| \leqq \|x-y\|$.

It holds that

$$\begin{aligned} \|x\| &\leqq \|x-y+y\| \\ &\leqq \|x-y\| + \|y\|. \end{aligned} \quad \downarrow (N3)$$

Thus, $\|x\| - \|y\| \leqq \|x-y\|$.

Also, $$\begin{aligned} \|y\| &= \|y-x+x\| \\ &\leqq \|y-x\| + \|x\| \quad \downarrow (N3) \\ &= \|-(x-y)\| + \|x\| \quad \downarrow (N2) \\ &= |-1| \|x-y\| + \|x\| \\ &= \|x-y\| + \|x\|. \end{aligned}$$

Therefore, $-\|x-y\| \leqq \|x\| - \|y\|$.

//

$$\boxed{\begin{array}{l} E \text{ normed space} \\ \{x_n\} \subset E \\ x \in E \\ x_n \to x \\ \Rightarrow \|x_n\| \to \|x\| \end{array}}$$

<u>Proof.</u>

It follows that

$$\big| \|x_n\| - \|x\| \big| \leq \|x_n - x\| \to 0.$$

//

"continuity of $\|\cdot\|$"

$$\|x_n\| \to \|x\|$$

$$\Leftrightarrow \|x\| = \lim \|x_n\|$$

$$\|\lim x_n\|$$

$$\Leftrightarrow \lim \|x_n\| = \|\lim x_n\|$$

$$\boxed{\begin{aligned} & x_n \to x \\ & y_n \to y \\ & \Rightarrow x_n + y_n \to x + y \end{aligned}}$$

Proof

It holds true that

$$\|(x_n+y_n)-(x+y)\|$$

$$= \|(x_n - x) + (y_n - y)\|$$

$$\leq \|x_n - x\| + \|y_n - y\|$$

$$\to 0. \quad //$$

"continuity of sum"

$\{\alpha_n\} \subset K,\ \alpha \in K$

$x_n \to x$

$\alpha_n \to \alpha$

$\Longrightarrow \alpha_n x_n \to \alpha x$

<u>Proof</u>

The following is obtained:

$$
\begin{aligned}
&\|\alpha_n x_n - \alpha x\| \\
&\leq \|\alpha_n x_n - \alpha x_n\| + \|\alpha x_n - \alpha x\| \\
&= \|(\alpha_n - \alpha) x_n\| + \|\alpha(x_n - x)\| \\
&= \underbrace{|\alpha_n - \alpha|}_{\to 0}\ \underbrace{\|x_n\|}_{\to \|x\|} + |\alpha|\ \underbrace{\|x_n - x\|}_{\to 0} \\
&\to 0.
\end{aligned}
$$

//

Cor

E K-normed space
$\{x_n\} \subset E$, $x \in E$
$\alpha \in K$
$\Rightarrow \alpha x_n \to \alpha x$

"continuity of scalar multiplication"

Cor

$x_n \to x$
$y_n \to y$
$\alpha, \beta \in K$
$\Rightarrow \alpha x_n + \beta y_n \to \alpha x + \beta y$

Normed spaces

1. ベクトル空間Vとその要素x,yについて，$x-y=-(y-x)$となることを証明せよ.

2. ノルム空間の定義を述べ，ノルム空間$(E,\|\cdot\|)$が与えられたとき，$d(x,y)=\|x-y\|$と定義すると(E,d)が距離空間になることを証明せよ.

3. 実ベクトル空間$\mathbb{R}^2$において，$\|(x,y)\|_1=|x|+|y|$と定義すると，$(\mathbb{R}^2,\|\cdot\|_1)$はノルム空間になることを証明せよ.

4. 実ベクトル空間$B(X)=\{f:X\to\mathbb{R}\mid f\text{ is bounded.}\}$において，$\|f\|_\infty=\sup_{x\in X}|f(x)|$と定義すると，$(B(X),\|\cdot\|_\infty)$はノルム空間になることを証明せよ．また，実ベクトル空間$L(\mathbb{R})=\{f\mid f:\mathbb{R}\to\mathbb{R}\}$にこのような形ではノルムを入れにくい理由は何か？
※このノルムをsup-normと呼ぶこともある.

5. コンパクト距離空間X上で定義される連続関数の集合$C(X)=\{f:X\to\mathbb{R}\mid f\text{ is continuous.}\}$は，有界関数がなす実ノルム空間$B(X)$の部分空間なので，問題4で導入された$B(X)$のノルム(sup-norm)$\|\cdot\|_\infty$を$C(X)$に制限して考えるとそれ自体が実ノルム空間になる.
(1) $C^1(X)=\{f:X\to\mathbb{R}\mid f\text{ is continuously differentiable.}\}$が，$C(X)$の部分空間になることを確認せよ.
(2) $C^1(X)$に属す実数値関数fに対して
$$\|f\|=\|f\|_\infty+\|f'\|_\infty$$
と定義すると，これは$C^1(X)$上のノルムになる．このことを証明せよ.

6. ノルム空間$(E,\|\cdot\|)$において，不等式$|\|x\|-\|y\||\leq\|x-y\|$を証明せよ．その際，ノルムの公理の何をどこで用いたかを指摘せよ.

7. $\{x_n\},\{y_n\}$をノルム空間Eにおける点列，$\{\alpha_n\}$をスカラー列，$x,y\in E$，αをスカラーとする．$x_n\to x$，$y_n\to y$，$\alpha_n\to\alpha$とすると，以下が成り立つことを示せ.
(1) $\|x_n\|\to\|x\|$,　(2) $x_n+y_n\to x+y$,　(3) $\alpha_n x_n\to\alpha x$

8. 問題7の(2)と(3)によると，点列に対してその極限を対応させる写像は線型性を持っているといえそうである．適当なベクトル空間を設定して，このことを正確に述べてみよ.

9. ノルム空間$(E,\|\cdot\|)$における閉単位球$\bar{S}=\{x\in E\mid\|x\|\leq 1\}$が閉凸集合であることを証明せよ.