Banach spaces

__Def.__

E <u>$\mathbb{K}$-Banach space</u>

($\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$)

$\Leftrightarrow$ (A) E $\mathbb{K}$-normed space

(B) E : complete

E K-Banach space

Then,

(I) E : K-vector space

$\downarrow$

$x+y$, αx

- $\exists^1 0 \in E$: $\forall x \in E$, $x+0=0+x=x$.
- $\forall x \in E$, $\exists^1 -x \in E$; $x+(-x)=(-x)+x=0$

$\vdots$

(II) E : K-normed space

$\|x\|$

$d(x,y)=\|x-y\|$

$\Rightarrow (E,d)$ M.S

$\downarrow$

$S_r(x)$ open sphere

$\{x_n\} \subset E$ Cauchy sequence

(III) (E,d) : complete

ex

$\mathbb{R}$ with $|\cdot|$

$\Rightarrow (\mathbb{R}, |\cdot|)$ $\mathbb{R}$-Banach space

ex

$X \neq \emptyset$

$B(X) = \{ f : X \to \mathbb{R} \mid f \text{ is bdd.} \}$

$\|f\| = \sup_{x \in X} |f(x)| \quad \forall f \in B(X)$

$\Rightarrow (B(X), \|\cdot\|)$ $\mathbb{R}$-Banach space

Th

$X \neq \emptyset$

$B(X) = \{ f : X \to \mathbb{R} \mid f \text{ is bdd.} \}$

$d(f, g) = \sup_{x \in X} |f(x) - g(x)| \quad \forall f, g \in B(X)$

$\Rightarrow (B(X), d)$: CMS

- $X = [a, b]$ $(\subset \mathbb{R})$

  $B([a,b])$

  $$\|f\| = \sup_{x \in [a,b]} |f(x)|$$

- $X = (a, b]$ $(\subset \mathbb{R})$

- $X = (a, \infty)$ $(\subset \mathbb{R})$

- $X = \mathbb{N}$

  $B(\mathbb{N}) = \{ x = \{x_n\} \subset \mathbb{R} \mid \{x_n\} \text{ is bdd.} \}$

  $$\|x\| = \sup_{n \in \mathbb{N}} |x_n|$$

- $X = \{1, 2\}$

  $\mathbb{R}^2$

  $$\|(x_1, x_2)\|_\infty = \max\{|x_1|, |x_2|\}$$

Set of sequences

■ $L(\mathbb{N}) = \{ x = \{x_n\} \mid \forall n \in \mathbb{N}, x_n \in \mathbb{R} \}$

- Equality

  $\{x_n\} = \{y_n\}$

  $\Leftrightarrow \forall n \in \mathbb{N}, x_n = y_n$

- $L(\mathbb{N})$: $\mathbb{R}$-vector space

  - $\{x_n\} + \{y_n\} = \{x_n + y_n\}$
  - $\alpha\{x_n\} = \{\alpha x_n\}$

ex

$\{x_n\} = \{1, 2, 3, 4, \cdots\}$

$\{y_n\} = \{0, -1, 0, -1, \cdots\}$

Then,

$2\{x_n\} - \{y_n\}$

$= \{2, 5, 6, 9, \cdots\}$

- zero element

  $0 = \{0, 0, 0, \cdots\}$

- inverse element

  $-\{x_n\} = \{-x_1, -x_2, -x_3, \cdots\}$

$\boxed{\ } B(\mathbb{N}) = \{x = \{x_n\} \in L(\mathbb{N}) \mid \{x_n\} \text{ is bdd.}\}$

- norm (sup-norm)

$$\|x\| = \|\{x_n\}\|$$
$$= \sup_{n \in \mathbb{N}} |x_n| \quad \text{where } x = \{x_n\} \in B(\mathbb{N}).$$

- distance

$$d(x, y) = \|x - y\|$$
$$= \sup_{n \in \mathbb{N}} |x_n - y_n|$$

where $x = \{x_n\}, y = \{y_n\} \in B(\mathbb{N})$.

ex

$x = \{1, 2, 3, 1, 2, 3, \cdots\}$

$y = \{-1, -2, -3, -1, -2, -3, \cdots\}$

Then,

$$d(x, y) = \sup_{n \in \mathbb{N}} |x_n - y_n| = 6$$

◦ open sphere

$$S_r(x) = \{ y = \{y_n\} \in B(\mathbb{N}) \mid d(x, y) < r \}.$$

Let $\begin{cases} x = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\} \\ y = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\} \\ z = \{1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \cdots\} \end{cases}$

Then, $\begin{cases} y \in S_1(x) \\ z \notin S_1(x). \end{cases}$

◦ sequence in $B(\mathbb{N})$

$$x^{(1)} = \{x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \cdots\} \in B(\mathbb{N})$$

$$x^{(2)} = \{x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \cdots\} \in B(\mathbb{N})$$

$$\vdots$$

$\{x^{(n)}\}_{n=1}^{\infty} \subset B(\mathbb{N})$

sequence in $B(\mathbb{N})$.

∘ Cauchy sequence

$$d(x^{(m)}, x^{(n)}) \to 0 \quad (m, n \to \infty)$$

where $d(x^{(m)}, x^{(n)})$

$$= \sup_{i \in \mathbb{N}} |x_i^{(m)} - x_i^{(n)}|$$

$$x^{(1)} = \{x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \cdots\} \in B(\mathbb{N})$$

$$x^{(2)} = \{x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \cdots\} \in B(\mathbb{N})$$

$$\cdots\cdots$$

$$x^{(n)} = \{x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \cdots\} \in B(\mathbb{N})$$

$$\cdots\cdots$$

$$x^{(m)} = \{x_1^{(m)}, x_2^{(m)}, x_3^{(m)}, \cdots\} \in B(\mathbb{N})$$

$$\cdots\cdots$$

$$\downarrow$$

$$x = \{x_1, x_2, x_3, \cdots\}$$

As $B(\mathbb{N})$ is a $\mathbb{R}$-Banach space (complete),

$\{x^{(n)}\}_{n=1}^{\infty} \subset B(\mathbb{N})$: Cauchy sequence

$\Leftrightarrow {}^{\exists} x = \{x_n\} \in B(\mathbb{N}) : x^{(n)} \to x$.

$\boxed{\begin{array}{l} E \text{ normed space} \\ M \subset E \text{ subspace} \\ \Rightarrow \overline{M} : \text{subspace} \end{array}}$

Proof.

Let $x, y \in \overline{M}$; $\alpha, \beta \in K$.

i.e. $\begin{cases} \exists \{x_n\} \subset M : x_n \to x \\ \exists \{y_n\} \subset M : y_n \to y \end{cases}$ — (*)

We prove that $\alpha x + \beta y \in \overline{M}$.

i.e. $\exists \{z_n\} \subset M : z_n \to \alpha x + \beta y$

Define $z_n = \alpha x_n + \beta y_n$.

As $x_n, y_n \in M$ and M is a subspace,

we have that $z_n = \alpha x_n + \beta y_n \in M$.

From (*), we obtain

$$z_n = \alpha x_n + \beta y_n \to \alpha x + \beta y.$$

//

E Banach space
$M \subset E$ subspace
$\Rightarrow \overline{M}$: Banach space

Proof
As M is a subspace of $(E, \|\cdot\|)$,
$\overline{M}$ is also a subspace.
As E is complete and $\overline{M}$ is closed in E,
$\overline{M}$ is also complete.
Therefore, $(\overline{M}, \|\cdot\|)$ is a Banach space.
//

Th
X CMS
$A \subset X$
$\Rightarrow$ Equivalent
① A is complete
② A $(\subset X)$ is closed.

Def.

$X \neq \emptyset$

$f_n : X \to \mathbb{R} \quad (n \in \mathbb{N})$

$f : X \to \mathbb{R}$

- $f_n \to f$ pointwisely $(n \to \infty)$

$\iff {}^\forall x \in X,\ f_n(x) \to f(x) \quad (n \to \infty)$

- $f_n \to f$ uniformly $(n \to \infty)$

$\iff \sup_{x \in X} |f_n(x) - f(x)| \to 0 \quad (n \to \infty)$

各点収束と一様収束

- $f_n \to f$ pointwisely

$\Leftrightarrow \forall x \in X, \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}:$

$\forall n \geqq n_0, \ |f_n(x) - f(x)| < \varepsilon.$

- $f_n \to f$ uniformly

$\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}: \forall n \geqq n_0,$

$$\sup_{x \in X} |f_n(x) - f(x)| < \varepsilon$$

$\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}: \forall n \geqq n_0,$

$\forall x \in X, \ |f_n(x) - f(x)| < \varepsilon.$

$X \neq \emptyset$

$f_n, f : X \to \mathbb{R} \ (n \in \mathbb{N})$

$f_n \to f$ uniformly

$\Rightarrow f_n \to f$ pointwisely

Th

X MS

$f_n : X \to \mathbb{R}$ continuous $(n \in \mathbb{N})$

$f : X \to \mathbb{R}$

$f_n \to f$ uniformly — (*)

$\Rightarrow f$ is continuous

Proof

We prove that

$\forall x_0 \in X, \forall \varepsilon > 0, \exists \delta > 0:$

$$d(x, x_0) < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon .$$

Let $x_0 \in X$ and let $\varepsilon > 0$.

From (*), for $\varepsilon/3 > 0$,

$\exists n_0 \in \mathbb{N} : \forall n \geq n_0,$

$$\sup_{x \in X} |f_n(x) - f(x)| < \frac{\varepsilon}{3}. \quad \text{— (**)}$$

As f_{n_0} is continuous at x_0, for $\varepsilon/3 > 0$,

$\exists \delta > 0:$

$$d(x, x_0) < \delta \Rightarrow |f_{n_0}(x) - f_{n_0}(x_0)| < \frac{\varepsilon}{3}.$$

— (*3)

Let $x \in X : d(x, x_0) < \delta$.

Then, it follows that

$$
\begin{aligned}
&|f(x) - f(x_0)| \\
&\leq \underbrace{|f(x) - f_{n_0}(x)|}_{} + \underbrace{|f_{n_0}(x) - f_{n_0}(x_0)|}_{< \varepsilon/3 \ (*3)} \\
&\qquad + \underbrace{|f_{n_0}(x_0) - f(x_0)|}_{< \frac{\varepsilon}{3} \ (**)} \\
&< \varepsilon.
\end{aligned}
$$

$\therefore \forall x_0 \in X, \varepsilon > 0, \exists \delta > 0 :$

$$d(x, x_0) < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

$\therefore f$ is continuous on X. //

<u>ex</u>

$f_n : [0,1] \to \mathbb{R}$ defined as follows:

$$f_n(x) = x^n \quad x \in [0,1]$$

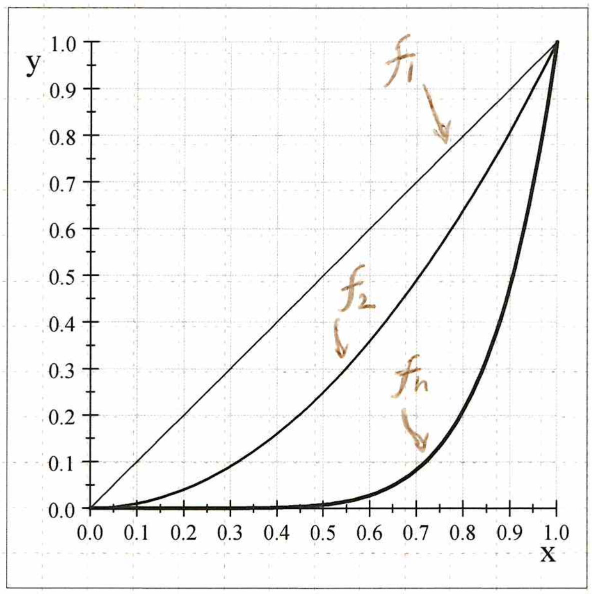
$f : [0,1] \to \mathbb{R}$ defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Then, $f_n \to f$ pointwisely.

As f is not continuous,

(f_n) does not converge uniformly.



X compact MS

$B(X) = \{f : X \to \mathbb{R} \mid f \text{ is bdd.}\}$

$\|f\| = \sup_{x \in X} |f(x)| \quad \forall f \in B(X)$

$C(X) = \{f : X \to \mathbb{R} \mid f \text{ is continuous.}\}$

$\Rightarrow C(X)$ is closed in $B(X)$.

Proof.

As X is compact, we have that

$C(X) \subset B(X)$.

Let $\{f_n\} \subset C(X) : f_n \to f \in B(X)$.

i.e. $\|f_n - f\| \to 0$

i.e. $\sup_{x \in X} |f_n(x) - f(x)| \to 0. \quad -(*)$

We show that $\underline{f \in C(X)}$.

Note that

$(*) \Leftrightarrow \{f_n\}$ converges uniformly to f.

As f_n is continuous $(n \in \mathbb{N})$,

f is also continuous.

i.e. $f \in C(X)$. //

Th

X compact MS

$C(X) = \{f : X \to \mathbb{R} \mid f \text{ is continuous.}\}$

$\|f\| = \sup_{x \in X} |f(x)| \quad \forall f \in C(X)$

$\Rightarrow (C(X), \|\cdot\|)$ $\mathbb{R}$-Banach space

Proof

$C(X)$ is a subspace of

$B(X) = \{f : X \to \mathbb{R} \mid f \text{ is bdd.}\}$.

As $B(X)$ is complete with the sup-norm

and $C(X)$ is closed in $B(X)$,

$C(X)$ is also complete.

Therefore, $C(X)$ is a Banach space.

//

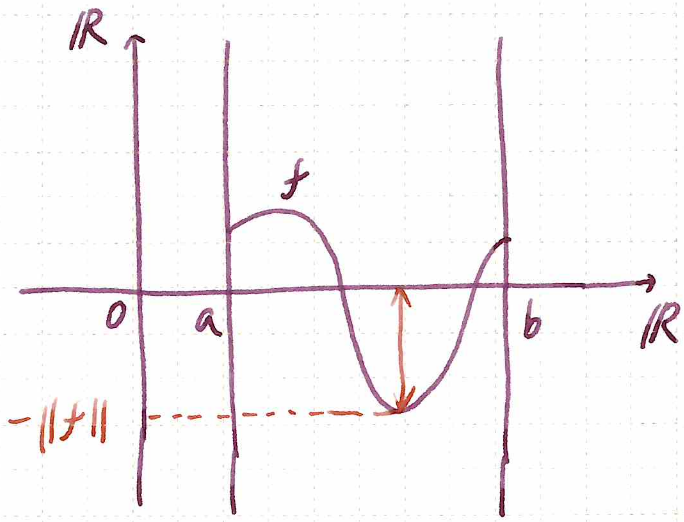
Cor

$C([a,b])$

$= \{ f:[a,b]\to\mathbb{R} \mid f \text{ is continuous.} \}$

$\|f\| = \sup_{x\in[a,b]} |f(x)| \quad \forall f\in C([a,b])$

$\Rightarrow (C([a,b]), \|\cdot\|)$ $\mathbb{R}$-Banach space



Th

X MS

$B(X) = \{f : X \to \mathbb{R} \mid f \text{ is bdd.}\}$

$\|f\| = \sup_{x \in X} |f(x)| \quad \forall f \in B(X)$

$BC(X) = \{f \in B(X) \mid f \text{ is continuous.}\}$

$\Rightarrow BC(X)$ is closed in $B(X)$.

Proof.

Let $\{f_n\} \subset BC(X) : f_n \to f \in B(X)$.

We prove that $f \in BC(X)$.

OK. //

Th

X MS

$BC(X) = \{ f : X \to \mathbb{R} \mid f$ is bdd and continuous. $\}$

$$\|f\| = \sup_{x \in X} |f(x)| \quad \forall f \in BC(X)$$

$\Rightarrow (BC(X), \|\cdot\|)$ $\mathbb{R}$-Banach space

Cor

$I \subset \mathbb{R}, \neq \emptyset$

$BC(I) = \{ f : I \to \mathbb{R} \mid f$ is bdd and continuous. $\}$

$$\|f\| = \sup_{x \in I} |f(x)| \quad \forall f \in BC(I)$$

$\Rightarrow (BC(I), \|\cdot\|)$ $\mathbb{R}$-Banach space

- $A, B \in \mathbb{R}$

  $\Rightarrow \frac{A+B}{2} \leqq \max\{A, B\}$

- $A, B \geqq 0$

  $\Rightarrow \max\{A, B\} \leqq A+B$

$$\boxed{\begin{array}{l} A, B \geqq 0 \\ \Rightarrow \max\{A, B\} \leqq A+B \leqq 2 \cdot \max\{A, B\} \end{array}}$$

$$\boxed{\begin{array}{l} \text{For } (x, y) \in \mathbb{R}^2, \\ \|(x, y)\|_\infty \leqq \|(x, y)\|_1 \leqq 2\|(x, y)\|_\infty \end{array}}$$

$\{x_n\} \subset \mathbb{R}^2$

$\Rightarrow$ Equivalent

① $\{x_n\}$: Cauchy seq. in $\|\cdot\|_\infty$

② $\{x_n\}$: Cauchy seq. in $\|\cdot\|_1$

$\{x_n\} \subset \mathbb{R}^2$, $x \in \mathbb{R}^2$

$\Rightarrow$ Equivalent

① $\|x_n - x\|_\infty \to 0$

② $\|x_n - x\|_1 \to 0$

Th

$(\mathbb{R}^2, \|\cdot\|_1)$ Banach space

Banach spaces

1. これまでわかっているバナッハ空間(Banach space)の例を挙げよ. また, 数列の集合で実バナッハ空間の例を挙げ, ベクトル空間としての構造やノルム, 距離について説明せよ. そこでの開球やコーシー列についても説明せよ.

2. ノルム空間$(E, \|\cdot\|)$の部分空間Mについて, その閉包$\overline{M}$も部分空間になることを示せ.

3. バナッハ空間$(E, \|\cdot\|)$の部分空間Mについて, その閉包$\overline{M}$もバナッハ空間になることを示せ.

4. 空でない集合X上で定義された実数値関数列$\{f_n\}$について各点収束と一様収束の定義を述べ, それらの違いを説明せよ.

5. 距離空間X上で定義された実数値連続関数列$\{f_n\}$について, その一様収束先の極限関数も連続であることを証明せよ.

6. 連続関数列がある関数に各点収束するとしても, 極限関数は連続とは限らない. そのことを示す例を挙げよ.

7. コンパクトな距離空間X上の実数値有界関数の集合を$B(X)$と表す. また, そこにsup-normを入れる.

$$C(X) = \{f : X \to \mathbb{R} \mid f \text{ is continuous.}\}$$

と表すと, $C(X)$は$B(X)$の部分集合であるが, さらに閉集合となることを示せ.

8. 問題7で導入したコンパクトな距離空間X上の連続関数の全体$C(X)$はsup-normを入れると, 実バナッハ空間になる. なぜか？

9. 距離空間X上の実数値**有界**連続関数の集合$BC(X)$にsup-normを入れると, $BC(X)$は実バナッハ空間になる. このことを証明せよ.

10. 次を確認していくことで, $(\mathbb{R}^2, \|\cdot\|_1)$がバナッハ空間になることを納得せよ.

(1) $A, B \in \mathbb{R}$に対して, $\frac{A+B}{2} \leq \max\{A, B\}$が成り立つ.

(2) $A, B \geq 0$に対して, $\max\{A, B\} \leq A + B$が成り立つ.

(3) $A, B \geq 0$に対して, $\max\{A, B\} \leq A + B \leq 2\max\{A, B\}$が成り立つ.

(4) $(x, y) \in \mathbb{R}^2$に対して, $\|(x, y)\|_\infty \leq \|(x, y)\|_1 \leq 2\|(x, y)\|_\infty$が成り立つ.

(5) $\{x_n\}$を$\mathbb{R}^2$の点列とすると, $\{x_n\}$が$(\mathbb{R}^2, \|\cdot\|_\infty)$においてコーシー列であることと, $(\mathbb{R}^2, \|\cdot\|_1)$においてコーシー列であることは同値である.

(6) $\{x_n\}$を$\mathbb{R}^2$の点列, また$x \in \mathbb{R}^2$とする. このとき, $\|x_n - x\|_\infty \to 0$と$\|x_n - x\|_1 \to 0$は同値である.

(7) 以上より, $(\mathbb{R}^2, \|\cdot\|_1)$はバナッハ空間である.

11. $(\mathbb{R}^3, \|\cdot\|_1)$はバナッハ空間になるだろうか？ 考えてみよ. ただしここで, $\|(x, y, z)\|_1 = |x| + |y| + |z|$である.

12. 改めてバナッハ空間の例を挙げよ.