Pre-Hilbert spaces

Complex numbers

Let $\begin{cases} x = a + bi \in \mathbb{C}, \\ y = c + di \in \mathbb{C}. \end{cases}$

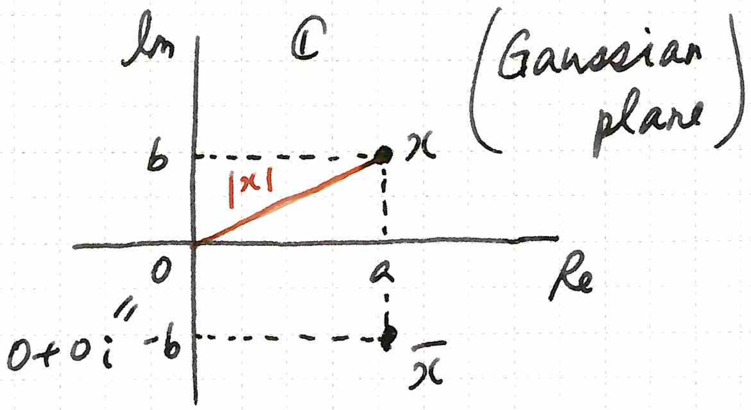
where $a, b, c, d \in \mathbb{R}$

$a = \mathrm{Re}\,x$, $b = \mathrm{Im}\,x$, $c = \mathrm{Re}\,y$, $d = \mathrm{Im}\,y$

- $\bar{x} = a - bi \in \mathbb{C}$

  complex conjugate 共役複素数

- $|x| = \sqrt{a^2 + b^2}$ absolute value



- $x+y$, $x-y$, xy, $\dfrac{x}{y}$ $(y \neq 0)$

① $\overline{\overline{x}} = x$

② $x + \overline{x} = 2\,\mathrm{Re}\,x \ (= 2a)$

③ $|x| = |\overline{x}| \ (= \sqrt{a^2+b^2})$

④ $x\overline{x} = |x|^2$

⑤ $|x|^2 \neq x^2$

check

④: It follows that

$$x\overline{x} = (a+bi)(a-bi)$$
$$= a^2+b^2 = |x|^2$$

⑤: On the one hand,

$$|x|^2 = a^2+b^2$$

On the other hand,

$$x^2 = (a+bi)(a+bi)$$
$$= a^2+2abi-b^2$$

Therefore, $|x|^2 \neq x^2$.

$$\boxed{\overline{x+y} = \bar{x} + \bar{y}}$$

($\because$)

On the one hand,

$$\overline{x+y} = \overline{(a+bi)+(c+di)}$$
$$= \overline{(a+c)+(b+d)i}$$
$$= (a+c)-(b+d)i.$$

On the other hand,

$$\bar{x}+\bar{y} = (a-bi)+(c-di)$$
$$= (a+c)-(b+d)i.$$

Therefore, we obtain the desired result.

//

$$\boxed{\overline{xy} = \overline{x}\,\overline{y}}$$

(i) On the one hand,

$$\overline{xy} = \overline{(a+bi)(c+di)}$$
$$= \overline{ac-bd+(ad+bc)i}$$
$$= ac-bd-(ad+bc)i.$$

On the other hand,

$$\overline{x}\,\overline{y} = (a-bi)(c-di)$$
$$= (ac-bd)-(ad+bc)i.$$

$$\therefore \overline{xy} = \overline{x}\,\overline{y}.$$

//

$$\boxed{\begin{array}{l} \overline{-x} = -\overline{x} \\ \overline{xyz} = \overline{x}\,\overline{y}\,\overline{z} \\ \overline{\left(\dfrac{x}{y}\right)} = \dfrac{\overline{x}}{\overline{y}} \quad \text{where } y \neq 0 \, (=0+0i) \end{array}}$$

Def

H *pre-Hilbert space*

$\Leftrightarrow$ (I) H : $\mathbb{C}$-vector space

(II) $\langle \cdot , \cdot \rangle : H \times H \to \mathbb{C}$

(I1) $\langle x, x \rangle \geqq 0$; $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

(I2) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(I3) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

(I4) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

Remark.

From (I4), $\langle x, x \rangle \in \mathbb{R}$.

Remark

Define $f : H \to \mathbb{C}$ as follows:

$$f(x) = \langle x, y \rangle \quad {}^{\forall} x \in H,$$

where $y \in H$ is fixed.

Then, from (I2) and (I3), f is linear.

H $\mathbb{C}$-pre-Hilbert space

$x, y, z \in H$

$b, c \in \mathbb{C}$

$\Rightarrow \langle x, by + cz \rangle = \bar{b} \langle x, y \rangle + \bar{c} \langle x, z \rangle$

<u>Proof</u>

$$\begin{aligned}
\text{LHS} &= \overline{\langle by + cz, x \rangle} && \leftarrow (I4) \\
&= \overline{\langle by, x \rangle + \langle cz, x \rangle} && (I2) \\
&= \overline{b \langle y, x \rangle + c \langle z, x \rangle} && (I3) \\
&= \overline{b \langle y, x \rangle} + \overline{c \langle z, x \rangle} \\
&= \bar{b}\, \overline{\langle y, x \rangle} + \bar{c}\, \overline{\langle z, x \rangle} \\
&= \bar{b}\, \overline{\overline{\langle x, y \rangle}} + \bar{c}\, \overline{\overline{\langle x, z \rangle}} && (I4) \\
&= \bar{b} \langle x, y \rangle + \bar{c} \langle x, z \rangle .
\end{aligned}$$

//

H pre-Hilbert space over $\mathbb{R}$
$x, y \in H$
$\Rightarrow$ Equivalent
① $x = y$
② $\langle x, z \rangle = \langle y, z \rangle \ \forall z \in H$
③ $\langle x-y, z \rangle \leqq 0 \ \forall z \in H$

Proof

①⇒② OK

②⇒③ OK

③⇒①

From ③, we have
$$\langle x-y, z \rangle \leqq 0 \ \forall z \in H.$$
Letting $z = x-y \in H$, we obtain
$$\langle x-y, x-y \rangle \leqq 0.$$
From (I1), $\langle x-y, x-y \rangle \geqq 0$.
$$\therefore \langle x-y, x-y \rangle = 0.$$
Using (I1) again, we have
$$x-y=0.$$
$\therefore x = y$.

//

ex

$H = \mathbb{R}$

$\langle x, y \rangle = xy$

ex

$H = \mathbb{C}$

$\langle x, y \rangle = x\bar{y}$

$\Rightarrow (H, \langle \cdot, \cdot \rangle)$ $\mathbb{C}$-pre-Hilbert space

Proof

(I1) $\langle x, x \rangle \geq 0$; $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

So $\langle x, x \rangle = x\bar{x} = |x|^2$, (I1) holds true.

(I2) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

$LHS = (x+y)\bar{z} = x\bar{z} + y\bar{z} = RHS$

(I3) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

$LHS = (\alpha x)\bar{y} = \alpha \cdot x\bar{y} = RHS$

(I4) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

$LHS = x\bar{y} = \overline{\bar{x}y} = \overline{\langle y, x \rangle} = RHS$.

//

ex

$$H = \mathbb{R}^2$$

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle = ac + bd$$

$\Rightarrow (H, \langle \cdot, \cdot \rangle)$ $\mathbb{R}$-pre-Hilbert space

Proof

(I1) $\langle x, x \rangle \geq 0$; $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

As $\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle = a^2 + b^2$, the desired result holds.

(I2) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

$$\begin{aligned}
LHS &= \left\langle \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}, \begin{pmatrix} e \\ f \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} a+c \\ b+d \end{pmatrix}, \begin{pmatrix} e \\ f \end{pmatrix} \right\rangle \\
&= (a+c)e + (b+d)f \\
&= ae + ce + bf + df \\
&= \left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} e \\ f \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} c \\ d \end{pmatrix}, \begin{pmatrix} e \\ f \end{pmatrix} \right\rangle \\
&= RHS.
\end{aligned}$$

(I3) $\underline{\langle \alpha x, y\rangle = \alpha\langle x, y\rangle}$

$$LHS = \left\langle \alpha\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle$$

$$= (\alpha a)c + (\alpha b)d$$

$$= \alpha ac + \alpha bd$$

$$= \alpha(ac + bd)$$

$$= \alpha\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle = RHS$$

(I4) $\underline{\langle x, y\rangle = \langle y, x\rangle}$

$$LHS = \left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle$$

$$= ac + bd$$

$$= ca + db$$

$$= \left\langle \begin{pmatrix} c \\ d \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle$$

$$= RHS.$$

//

ex

$H = \mathbb{R}^N$

$x = (x_1, \cdots, x_N)$

$y = (y_1, \cdots, y_N)$

$$\langle x, y \rangle = \sum_{n=1}^{N} x_n y_n$$

$\Rightarrow (\mathbb{R}^N, \langle \cdot, \cdot \rangle)$: $\mathbb{R}$-pre-Hilbert space

⟨Euclidean space⟩

ex

$H = \mathbb{C}^2$

$x = (a+bi, c+di)$

$y = (s+ti, u+vi)$

$$\langle x, y \rangle = (a+bi)\overline{(s+ti)} + (c+di)\overline{(u+vi)}$$

$\Rightarrow (\mathbb{C}^2, \langle \cdot, \cdot \rangle)$:

$\mathbb{C}$-pre-Hilbert space

ex

$H = \mathbb{R}^2$

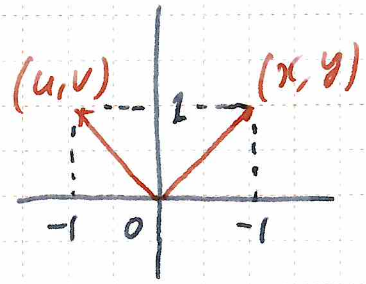
$(x, y), (u, v) \in \mathbb{R}^2$

$\langle (x, y), (u, v) \rangle = 2xu + yv$

$\Rightarrow (\mathbb{R}^2, \langle \cdot, \cdot \rangle)$ $\mathbb{R}$-pre-Hilbert space

$(x, y) = (1, 1)$

$(u, v) = (-1, 1)$



Then,

$$\begin{aligned}
&\langle (x, y), (u, v) \rangle \\
&= 2xu + yv \\
&= 2 \cdot 1 \cdot (-1) + 1 \cdot 1 \\
&= -2 + 1 \\
&= -1
\end{aligned}$$

$f \in C([a,b], \mathbb{R})$
$= \{g : [a,b] \to \mathbb{R} \mid g : \text{continuous.}\}$

$f \geqq 0$
$\Rightarrow$ Equivalent
① $f = 0$
② $\int_a^b f(x)\,dx = 0$

Proof

①⇒② Obvious.

②⇒①

Suppose by way of contradiction that
$\exists x_0 \in [a,b] : f(x_0) > 0.$

Then, $\exists \varepsilon > 0 : \forall x \in S_\varepsilon(x_0),\ f(x) > 0.$ — (*)

Let $\delta = \varepsilon/2 > 0.$

Then,

$S_\delta[x_0] = \{x \in [a,b] \mid |x - x_0| \leq \delta\} \subset S_\varepsilon(x_0).$

As $S_\delta[x_0]$ is compact,

$\exists x_* \in S_\delta[x_0] :$

$$f(x_*) = \inf_{z \in S_\delta[x_0]} f(z) = m.$$

So $x_* \in S_\delta[x_0] \subset S_\varepsilon(x_0)$,
we have from (*) that

$$f(x_*) = m > 0.$$

We obtain

$$0 = \int_a^b f(x)\,dx$$
$$= \int_{S_\delta[x_0]} f(x)\,dx + \int_{(S_\delta[x_0])^c} \underbrace{f(x)}_{\geq 0}\,dx$$
$$\geq \int_{S_\delta[x_0]} f(x)\,dx$$
$$\geq m \cdot 2\delta = m\varepsilon > 0.$$

This is a contradiction. //

ex

$$H = C([a,b], \mathbb{R})$$
$$= \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous.}\}$$
$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$$
$$\Rightarrow (H, \langle \cdot, \cdot \rangle) : \mathbb{R}\text{-pre-Hilbert space}$$

Proof

(I1) $\langle f, f \rangle \geqq 0$; $\langle f, f \rangle = 0 \Leftrightarrow f = 0$

(I2) $\langle f+g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$

$$LHS = \int_a^b (f(x) + g(x)) h(x)\, dx$$
$$= \int_a^b (f(x)h(x) + g(x)h(x))\, dx$$
$$= \int_a^b f(x)h(x)\, dx + \int_a^b g(x)h(x)\, dx$$
$$= \langle f, h \rangle + \langle g, h \rangle = RHS$$

(I3) $\langle \alpha f, g \rangle = \alpha \langle f, g \rangle$

(I4) $\langle f, g \rangle = \langle g, f \rangle$

//

$\underline{ex}$

$H_1 = C([0,1], \mathbb{R})$

$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx$

Let $\begin{cases} f(x) = x \\ g(x) = x^2. \end{cases}$

Then, $\langle f, g \rangle = \int_0^1 x^3 dx = \left[\frac{x^4}{4}\right]_0^1 = \underline{\frac{1}{4}}$

$\underline{ex}$

$H_2 = C([-1,1], \mathbb{R})$

$\langle f, g \rangle = \int_{-1}^1 f(x) g(x)\, dx$

Let $\begin{cases} f(x) = x \\ g(x) = x^2. \end{cases}$

Then, $\langle f, g \rangle = \int_{-1}^1 x^3 dx = \underline{0}$

※ H_1 and H_2 are different sets.
Thus, $(H_1, \langle \cdot, \cdot \rangle)$ and $(H_2, \langle \cdot, \cdot \rangle)$ are distinct pre-Hilbert space.
Consequently, different inner product values correspond to the same functions.
Strictly speaking, they should be regarded as different functions because their domains are different.

H $\mathbb{C}$-pre-Hilbert space

$\|x\| = \sqrt{\langle x, x\rangle}$

$\Rightarrow \|ax + by\|^2$

$= |a|^2\|x\|^2 + 2\,\mathrm{Re}(a\bar{b}\langle x, y\rangle) + |b|^2\|y\|^2$

where $x, y \in H$, $a, b \in \mathbb{C}$

Proof.

It follows that

$$
\begin{aligned}
\mathrm{LHS} &= \langle ax + by, ax + by\rangle \\
&= a\langle x, ax + by\rangle + b\langle y, ax + by\rangle \\
&= a(\bar{a}\langle x, x\rangle + \bar{b}\langle x, y\rangle) \\
&\quad + b(\bar{a}\langle y, x\rangle + \bar{b}\langle y, y\rangle) \\
&= a\bar{a}\|x\|^2 + a\bar{b}\langle x, y\rangle + \bar{a}b\overline{\langle x, y\rangle} + b\bar{b}\|y\|^2 \\
&= |a|^2\|x\|^2 + a\bar{b}\langle x, y\rangle + \overline{\overline{\bar{a}b\overline{\langle x, y\rangle}}} + |b|^2\|y\|^2 \\
&= |a|^2\|x\|^2 + a\bar{b}\langle x, y\rangle + \overline{a\bar{b}\langle x, y\rangle} + |b|^2\|y\|^2 \\
&= |a|^2\|x\|^2 + 2\,\mathrm{Re}(a\bar{a}\langle x, y\rangle) + |b|^2\|y\|^2. \quad //
\end{aligned}
$$

※ At this stage, we don't call $\|\cdot\|$ the norm.

Cor

H $\mathbb{R}$-pre-Hilbert space

$\|x\| \equiv \sqrt{\langle x, x \rangle}$

$\Rightarrow \|ax+by\|^2$

$= a^2\|x\|^2 + 2ab\langle x, y\rangle + b^2\|y\|^2$

- $a = b = 1$

  $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$

- $a = 1,\ b = -1$

  $\|x-y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$

Pre-Hilber spaces

1. 二つの複素数$x = a + bi,\ y = c + di$について，以下が成り立つことを確認せよ．

(1)$\overline{x+y} = \bar{x} + \bar{y}$,　(2)$\overline{xy} = \bar{x} \cdot \bar{y}$,　(3)$\overline{-x} = -\bar{x}$,　(4)$\overline{\left(\frac{x}{y}\right)} = \frac{\bar{x}}{\bar{y}}$.

2. 複素数$x = a + bi$の絶対値を$|x| = \sqrt{a^2 + b^2}$と定義する．このとき，(1)-(3)を確認せよ．ただし，$\bar{x} = a - bi$は$x = a + bi$の共役複素数である．

(1)$|x| = |\bar{x}|$,　(2)$x\bar{x} = |x|^2$,　(3)$x^2 \neq |x|^2$.

3. プレ・ヒルベルト空間（内積空間）の定義を確認し，以下の集合がプレ・ヒルベルト空間になるかチェックせよ．そうならないなら，プレ・ヒルベルト空間のどの条件が満たされないかを答えよ．

(1)集合$H = \mathbb{R}^2$において，$\langle (a,b),(c,d) \rangle = 2ac + bd$と定義．
(2)集合$H = \mathbb{R}^2$において，$\langle (a,b),(c,d) \rangle = ac$と定義．
(3)集合$H = \mathbb{R}^3$において，$\langle (x_1,x_2,x_3),(y_1,y_2,y_3) \rangle = x_1y_1 + x_2y_2 + x_3y_3$と定義．
(4)集合$H = \mathbb{C}$において，$\langle x,y \rangle = x\bar{y}$と定義．

4. 複素プレ・ヒルベルト空間Hにおいて，

$$\langle x,\ by + cz \rangle = \bar{b}\langle x,\ y \rangle + \bar{c}\langle x,\ z \rangle$$

が成り立つことを証明せよ．ただし$x,y,z \in H,\ b,c \in \mathbb{C}$である．

5. x,yを 実プレ・ヒルベルト空間Hの要素とする．このとき，次が同値であることを証明せよ．

(1)$x = y$,
(2)$\langle x,z \rangle = \langle y,z \rangle \quad \forall z \in H$,
(3)$\langle x - y,\ z \rangle \leq 0 \quad \forall z \in H$.

6. 集合$C([a,b])=\{f:\ [a,b] \to \mathbb{R} \mid f \text{ is continuous.}\}$の要素$f$について，$f \geq 0$とする．このとき，次の(1)と(2)が同値であることを示せ．

(1)$f = 0$,　(2)$\int_a^b f(x)dx = 0$.

7. 集合$C([a,b])=\{f:\ [a,b] \to \mathbb{R} \mid f \text{ is continuous.}\}$において，$\langle f,g \rangle = \int_a^b f(x)g(x)dx$と定義すると，これは$C([a,b])$上の内積になる．このことを証明せよ．また，ここで関数族の範囲を連続関数に限定している理由と定義域を有界閉集合$[a,b]$にしている理由を考察せよ．

8. 実プレ・ヒルベルト空間$C([-1,1])=\{f:\ [-1,1] \to \mathbb{R} \mid f \text{ is continuous.}\}$において，$f(x) = x,\ g(x) = -x$とする．内積$\langle f,g \rangle$を計算せよ．

9. 複素プレ・ヒルベルト空間において，$\|x\| = \sqrt{\langle x,x \rangle}$と記号を定義すると，

$$\|ax + by\|^2 = |a|^2\|x\|^2 + 2\,\mathrm{Re}\,a\bar{b}\langle x,y \rangle + |b|^2\|y\|^2$$

が成り立つ．このことを証明せよ．同様に，実プレ・ヒルベルト空間の場合について，

$$\|ax + by\|^2 = a^2\|x\|^2 + 2ab\langle x,y \rangle + b^2\|y\|^2$$

を証明せよ．

※次節では，$\|\cdot\|$がノルムの3条件を満たすことをシュワルツの不等式を用いて証明するが，この段階ではまだ未証明なので，$\|\cdot\|$をノルムと呼んでいない．