Series in normed spaces

E normed space

$\{x_n\} \subset E$

$\sum_{n=1}^{\infty} x_n$ series 級数

$s_n = \sum_{k=1}^{n} x_k$ partial sum 部分和

Def

- $\sum_{n=1}^{\infty} x_n$: convergent

  $\Leftrightarrow \{s_n\}$: convergent.

- $x = \sum_{n=1}^{\infty} x_n$

  $\Leftrightarrow s_n = \sum_{k=1}^{n} x_k \to x$

$x_1, x_2, x_3, \cdots$ sequence

$x_1 + x_2 + x_3 + \cdots$ series

$$\underbrace{\underbrace{\underbrace{\overset{s_1}{\overset{\shortparallel}{x_1}} + x_2}_{=s_2} + x_3}_{=s_3} + x_4}_{=s_4} + \cdots$$

$s_1, s_2, s_3, \cdots$ sequence

Remark

$$\boxed{x_n = s_n - s_{n-1}}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n} = \infty}$$

<u>Proof.</u>

It holds that

$$n \leqq x \iff \frac{1}{n} \geqq \frac{1}{x} \quad {}^{\forall} n \in \mathbb{N}.$$

Using this, we have

$$\frac{1}{n} = \int_n^{n+1} \frac{1}{n} dx$$

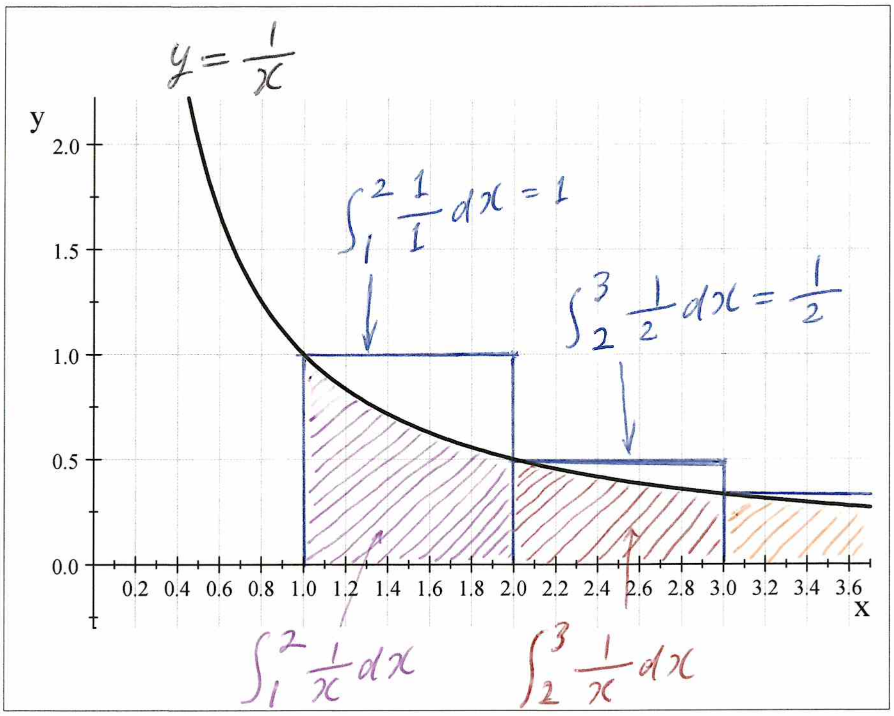
$$\geqq \int_n^{n+1} \frac{1}{x} dx$$

Summing these inequalities w.r.t. $n = 1, 2, \cdots, N-1$, we obtain

$$\sum_{n=1}^{N-1} \frac{1}{n} \geqq \int_1^N \frac{1}{x} dx$$

$$= [\log x]_1^N = \log N \quad {}^{\forall} N \in \mathbb{N}.$$

As $N \to \infty$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. //



$$\int_1^2 \frac{1}{1}\,dx + \int_2^3 \frac{1}{2}\,dx + \cdots + \int_{N-1}^{N} \frac{1}{N-1}\,dx$$

$$\geqq \int_1^2 \frac{1}{x}\,dx + \int_2^3 \frac{1}{x}\,dx + \cdots + \int_{N-1}^{N} \frac{1}{x}\,dx$$

$$= \int_1^N \frac{1}{x}\,dx$$

$$\therefore\ 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N-1}$$

$$\geqq \int_1^N \frac{1}{x}\,dx = \left[\log x\right]_1^N = \log N \quad (\forall N \in \mathbb{N})$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n} = \infty}$$

<u>Alternative proof.</u>

We show that

$$\underline{\sum_{n=1}^{2^N} \frac{1}{n} \geqq 1 + \frac{N}{2} \quad \forall N \in \mathbb{N} \cup \{0\}.}$$

using mathematical induction.

(i) If $N=0$, then the inequality holds.

(ii) Assume that $\displaystyle\sum_{n=1}^{2^k} \frac{1}{n} \geqq 1 + \frac{k}{2}$.

Then,

$$\sum_{n=1}^{2^k} \frac{1}{n} = \sum_{n=1}^{2 \cdot 2^k} \frac{1}{n}$$

$$= \sum_{n=1}^{2^k} \frac{1}{n} + \sum_{n=2^k+1}^{2 \cdot 2^k} \frac{1}{n}$$

$$\geqq 1 + \frac{k}{2} + \underbrace{\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2 \cdot 2^k}}_{2^k \text{ terms}}$$

$$\geqq 1 + \frac{k}{2} + 2^k \cdot \frac{1}{2 \cdot 2^k}$$

$$= 1 + \frac{k+1}{2}.$$

//

cf.

$$\boxed{\begin{array}{l}\{x_n\} \subset X \text{ Cauchy sequence} \\ \text{i.e. } d(x_m, x_n) \to 0 \ (m, n \to \infty) \\ \Rightarrow d(x_n, x_{n+1}) \to 0\end{array}}$$

$\not\Leftarrow$

ex

Let $\begin{cases} X = \mathbb{R} \\ x_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}. \end{cases}$

Then, $x_{n+1} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1}$.

Therefore,

$$d(x_n, x_{n+1}) = |x_n - x_{n+1}| = \frac{1}{n+1} \to 0.$$

However, for $m > n$,

$$d(x_n, x_m) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m}$$

$\to \infty$ as $m \to \infty$

with fixed $n \in \mathbb{N}$.

$\therefore d(x_n, x_m) \not\to 0$ as $m, n \to \infty$.

$$\boxed{\begin{array}{l} E \text{ normed space} \\ \{x_n\}, \{y_n\} \subset E \\ x = \sum_{n=1}^{\infty} x_n, \; y = \sum_{n=1}^{\infty} y_n \in E \\ \Rightarrow \sum_{n=1}^{\infty} (x_n + y_n) = x + y \end{array}} \quad - (*)$$

Proof.

Define $s_n = \sum_{k=1}^{n} x_k$, $t_n = \sum_{k=1}^{n} y_k \in E$.

From (*), $\begin{cases} s_n \to x \\ t_n \to y. \end{cases}$ — (**)

Our aim is to prove that

$$\sum_{k=1}^{n} (x_k + y_k) \to x + y.$$

As $\sum_{k=1}^{n} (x_k + y_k) = \sum_{k=1}^{n} x_k + \sum_{k=1}^{n} y_k$

$= s_n + t_n$,

we have from (**) that

the desired result follows. //

E K-normed space

$\{x_n\} \subset E$

$a \in K$

$$x = \sum_{n=1}^{\infty} x_n \in E \qquad — (*)$$

$$\Rightarrow \sum_{n=1}^{\infty} a x_n = a x$$

<u>Proof</u>

Define $S_n = \sum_{k=1}^{n} x_k \in E$.

From $(*)$, $S_n \to x$. $\qquad — (**)$

We show that $\underline{\sum_{k=1}^{n} a x_k = a x}$.

As $\sum_{k=1}^{n} a x_k = a \sum_{k=1}^{n} x_k = a S_n$,

from $(**)$, the desired result follows. //

$$\boxed{\begin{array}{l} E \text{ normed space} \\ \{x_n\} \subset E \\ \sum_{n=1}^{\infty} x_n \text{ is convergent in } E. \\ \Rightarrow x_n \to 0 \ (\in E) \end{array}} \quad \text{—} (*)$$

Proof.

Define $S_n \equiv \sum_{k=1}^{n} x_k \in E.$

Then, $x_n = S_n - S_{n-1} \ (\in E).$

From $(*)$, $\{S_n\}$ is convergent in E.

Thus, it is a Cauchy sequence.

We obtain

$$\|x_n\| = \|S_n - S_{n-1}\| \to 0.$$

//

$(\not\Leftarrow)$

ex

$E = \mathbb{R}$

$\{x_n\} = \left\{\frac{1}{n}\right\}.$

E normed space

$\{x_n\} \subset E$

$\Rightarrow$ Equivalent

① $\sum_{n=1}^{\infty} \|x_n\|$ is convergent.

② $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

Proof.

Define $r_n \equiv \sum_{k=1}^{n} \|x_k\| \in \mathbb{R}$.

Then, $\{r_n\} (\subset \mathbb{R})$

is monotone increasing.

Therefore, $r_n \to r \in (-\infty, \infty]$.

It follows that

① $\Leftrightarrow$ $\{r_n\} (\subset \mathbb{R})$ is convergent

$\Leftrightarrow$ $r \in \mathbb{R}$

$\Leftrightarrow$ ②. //

E Banach space
$\{x_n\} \subset E$
$\sum_{n=1}^{\infty} \|x_n\| < \infty$
$\Rightarrow \sum_{n=1}^{\infty} x_n$ is convergent.

Proof.

Define
$$\begin{cases} s_n = \sum_{n=1}^{n} x_k \in E, \\ r_n = \sum_{k=1}^{n} \|x_k\| \in \mathbb{R}. \end{cases}$$

As E is complete, it is sufficient to prove that $\{s_n\} (\subset E)$ is a Cauchy sequence.

Let $\varepsilon > 0$.

As $\{r_n\} (\subset \mathbb{R})$ is convergent, it is a Cauchy sequence.

For $\varepsilon > 0$, $\exists n_0 \in \mathbb{N} : \forall m, n \in \mathbb{N} : m \geq n$,
$$|r_m - r_n| = \|x_{n+1}\| + \cdots + \|x_m\| < \varepsilon.$$

Therefore, it holds that
$$\|s_m - s_n\| = \|x_{n+1} + \cdots + x_m\|$$
$$\leq \|x_{n+1}\| + \cdots + \|x_m\| < \varepsilon.$$
//

($\not\Leftarrow$)

ex

$E = \mathbb{R}$

$x_n = (-1)^{n-1} \frac{1}{n}$

Then, $\sum_{n=1}^{\infty} x_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

$= \log 2 < \infty.$

However, $\sum_{n=1}^{\infty} |x_n| = \infty.$

cf. $\log(1+x)$

$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$

$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad (-1 < x \leqq 1)$

$\downarrow x = 1$

$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Series in normed spaces

1. 級数とその収束について，定義を説明せよ．

2. $\sum_{n=1}^{\infty} \dfrac{1}{n} = \infty$を証明せよ．

3. 距離空間における点列$\{x_n\}$がコーシー列ならば，$d(x_n, x_{n+1}) \to 0$となるが，逆は言えないことを示す例を挙げよ．

4. $\{x_n\}$をノルム空間Eの点列とする．級数$\sum_{n=1}^{\infty} x_n$が収束するならば，$x_n \to 0$である．このことを示せ．また，逆は言えないことを示す例を挙げよ．

5. $\{x_n\}$をノルム空間Eの点列とする．このとき，次の2条件は同値である．そのことを証明せよ．

(1) $\sum_{n=1}^{\infty} \|x_n\|$は収束する．

(2) $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

6. $\{x_n\}$をバナッハ空間Eの点列とする．このとき，$\sum_{n=1}^{\infty} \|x_n\| < \infty$を仮定すると$\sum_{n=1}^{\infty} x_n$は収束する．これを証明せよ．また，逆は言えないことを示す例を挙げよ．