Angle, orthogonality, product spaces

Angle 角度

H $\mathbb{R}$-pre-Hilbert space

$x, y \in H$

$|\langle x, y\rangle| \leq \|x\|\|y\|$

⟨Schwarz's inequality⟩

$$\Big\downarrow\ x, y \neq 0$$

$$-1 \leq \frac{\langle x, y\rangle}{\|x\|\|y\|} \leq 1.$$

$$\boxed{\begin{aligned}&{}^{\forall}x, y \in H : x, y \neq 0,\\ &\quad {}^{\exists 1}\theta \in [0, \pi] :\\ &\qquad \cos\theta = \frac{\langle x, y\rangle}{\|x\|\|y\|}\end{aligned}}$$

$$\therefore \theta = \mathrm{Cos}^{-1}\left(\frac{\langle x, y\rangle}{\|x\|\|y\|}\right)$$

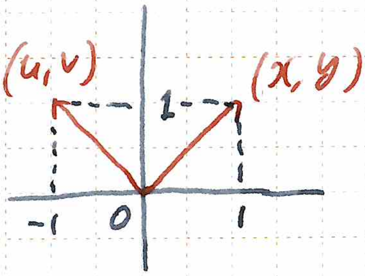
アーク・コサイン
$y = \cos x$ の
逆関数

ex

$H = \mathbb{R}^2$

$(x, y) = (1, 1)$

$(u, v) = (-1, 1)$



- $\langle (x,y), (u,v) \rangle = xu + yv$

Then, $\langle (x,y), (u,v) \rangle = 0$

$\therefore \cos\theta = 0 \quad \therefore \theta = \dfrac{\pi}{2} \fallingdotseq 1.5708\cdots$

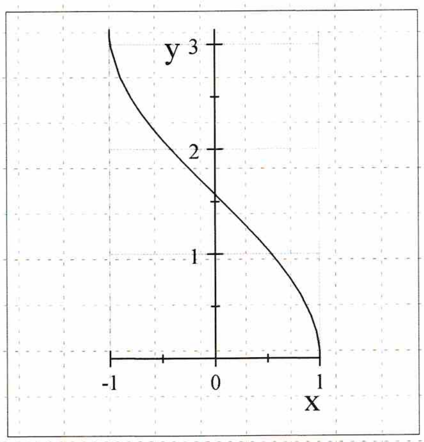
- $\langle (x,y), (u,v) \rangle = 2xu + yv$

Then, $\|(x,y)\| = \|(u,v)\| = \sqrt{3}$,

$\langle (x,y), (u,v) \rangle = -1$.

$\therefore \cos\theta = -\dfrac{1}{3} \quad \therefore \theta \fallingdotseq 1.9106\cdots$

$y = \arccos x$



Orthogonality 直交性

H $\mathbb{R}$-pre-Hilbert space

$x, y \in H$ are <u>orthogonal</u>

$\Leftrightarrow \langle x, y \rangle = 0$

$\Leftrightarrow x \perp y$

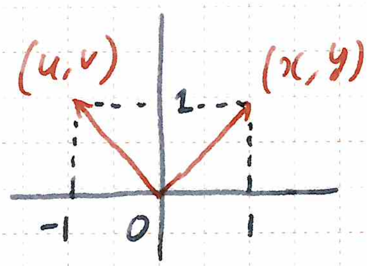
<u>Remark.</u> $\forall x \in H,\ x \perp 0$

<u>ex</u>

$H = \mathbb{R}^2$

$(x, y) = (1, 1)$

$(u, v) = (-1, 1)$



- <u>$\langle (x, y), (u, v) \rangle = xu + yv$</u>

  Then, $(x, y) \perp (u, v)$

- <u>$\langle (x, y), (u, v) \rangle = 2xu + yv$</u>

  Then, $(x, y) \not\perp (u, v)$

  not orthogonal

ex

$$\boxed{\begin{array}{l} H = C([-1,1], \mathbb{R}) \\ \langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \, dx \\ \Rightarrow (H, \langle \cdot, \cdot \rangle) : \mathbb{R}\text{-pre-Hilbert space} \end{array}}$$

Let $\begin{cases} f(x) = x \\ g(x) = x^2 . \end{cases}$

Then,

$$\langle f, g \rangle = \int_{-1}^{1} x^3 \, dx = 0 .$$

$\therefore f \perp g .$

V_1, V_2 K-vector spaces

$x=(x_1,x_2),\ y=(y_1,y_2)\in V_1\times V_2$

$x+y=(x_1+y_1,\ x_2+y_2)$

$\alpha x=(\alpha x_1,\alpha x_2)$, where $\alpha\in K$

$\Longrightarrow V\equiv V_1\times V_2$ K-vector space

<u>Proof</u>

(V1) $\underline{(x+y)+z=x+(y+z)}$

Let $z=(z_1,z_2)\in V_1\times V_2\equiv V$.

It follows that

$$\begin{aligned}
LHS &= \big((x_1,x_2)+(y_1,y_2)\big)+(z_1,z_2)\\
&=(x_1+y_1,\ x_2+y_2)+(z_1,z_2)\\
&=\big((x_1+y_1)+z_1,\ (x_2+y_2)+z_2\big)\\
&=\big(x_1+(y_1+z_1),\ x_2+(y_2+z_2)\big)\\
&=(x_1,x_2)+(y_1+z_1,\ y_2+z_2)\\
&=(x_1,x_2)+\big((y_1,y_2)+(z_1,z_2)\big)\\
&=x+(y+z)=RHS.
\end{aligned}$$

┘

(V2) $\exists 0 \in V : \forall x \in V,\ x + 0 = x$

Define $0 = (0, 0) \in V_1 \times V_2 \equiv V$.

Let $x = (x_1, x_2) \in V_1 \times V_2 \equiv V$.

It holds that

$$\begin{aligned} x + 0 &= (x_1, x_2) + (0, 0) \\ &= (x_1 + 0, x_2 + 0) \\ &= (x_1, x_2) = x. \end{aligned}$$

(V3) $\forall x \in V,\ \exists -x \in V : x + (-x) = 0.$

Let $x = (x_1, x_2) \in V_1 \times V_2 \equiv V$.

Define $-x = (-x_1, -x_2) \in V_1 \times V_2 \equiv V$,

where $-x_1$ and $-x_2$ are the inverse elements of x_1 and x_2, respectively.

Then, we have

$$\begin{aligned} x + (-x) &= (x_1, x_2) + (-x_1, -x_2) \\ &= (x_1 + (-x_1), x_2 + (-x_2)) \\ &= (0, 0) = 0. \end{aligned}$$

The parts after this are omitted. //

$$\left.\begin{array}{l}(H_1, \langle\cdot,\cdot\rangle_1) \\ (H_2, \langle\cdot,\cdot\rangle_2)\end{array}\right)\ \mathbb{C}\text{-pre-Hilbert spaces}$$

$x=(x_1,x_2),\ y=(y_1,y_2) \in H_1 \times H_2 \equiv H$

$\langle x,y\rangle \equiv \langle x_1,y_1\rangle_1 + \langle x_2,y_2\rangle_2$

$\Rightarrow (H,\langle\cdot,\cdot\rangle)$: $\mathbb{C}$-pre-Hilbert space

Proof.

(I1) $\underline{\langle x,x\rangle \geqq 0;\ \langle x,x\rangle = 0 \Leftrightarrow x=0\in H.}$

First, note that

$$\langle x,x\rangle = \langle (x_1,x_2),(x_1,x_2)\rangle$$
$$= \underbrace{\langle x_1,x_1\rangle_1}_{\geqq 0} + \underbrace{\langle x_2,x_2\rangle_2}_{\geqq 0} \geqq 0.$$

Furthermore,

$$\langle x,x\rangle = 0 \Leftrightarrow \left(\begin{array}{l}\langle x_1,x_1\rangle_1 \geqq 0 \\ \langle x_2,x_2\rangle_2 \geqq 0\end{array}\right.$$

$$\Leftrightarrow \left(\begin{array}{l}x_1 = 0 \in H_1 \\ x_2 = 0 \in H_2\end{array}\right.$$

$$\Leftrightarrow x = (x_1,x_2) = (0,0) = 0 \in H. \quad \lrcorner$$

(I2) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

Let $z = (z_1, z_2) \in H_1 \times H_2 = H$.

We have the following:

$$
\begin{aligned}
&\langle x+y, z\rangle \\
&= \langle (x_1, x_2) + (y_1, y_2), (z_1, z_2)\rangle \\
&= \langle (x_1+y_1, x_2+y_2), (z_1, z_2)\rangle \\
&= \langle x_1+y_1, z_1\rangle_1 + \langle x_2+y_2, z_2\rangle_2 \\
&= \langle x_1, z_1\rangle_1 + \langle y_1, z_1\rangle_1 \\
&\qquad + \langle x_2, z_2\rangle_2 + \langle y_2, z_2\rangle_2 \\
&= \langle (x_1, x_2), (z_1, z_2)\rangle \\
&\qquad + \langle (y_1, y_2), (z_1, z_2)\rangle \\
&= \langle x, z\rangle + \langle y, z\rangle. \quad \lrcorner
\end{aligned}
$$

(I3) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$

The proof is omitted here. $\lrcorner$

(I4) $\langle x, y\rangle = \overline{\langle y, x\rangle}$

It holds true that

$$\langle x, y\rangle = \langle (x_1, x_2), (y_1, y_2)\rangle$$
$$= \langle x_1, y_1\rangle_1 + \langle x_2, y_2\rangle_2$$
$$= \overline{\langle y_1, x_1\rangle_1} + \overline{\langle y_2, x_2\rangle_2}$$
$$= \overline{\langle y_1, x_1\rangle_1 + \langle y_2, x_2\rangle_2}$$
$$= \overline{\langle (y_1, y_2), (x_1, x_2)\rangle}$$
$$= \overline{\langle y, x\rangle}.$$

//

※ H is the product space of H_1 and H_2.

※ $$\|x\| = \sqrt{\langle x, x\rangle}$$
$$= \sqrt{\langle (x_1, x_2), (x_1, x_2)\rangle}$$
$$= \sqrt{\langle x_1, x_1\rangle_1 + \langle x_2, x_2\rangle_2}$$

$$\boxed{\therefore \|x\| = \sqrt{\|x_1\|_1^2 + \|x_2\|_2^2}}$$

ex

- $H_1 = \mathbb{R}$

$$\langle x_1, y_1 \rangle_1 = x_1 y_1$$

- $H_2 = \mathbb{R}^2$

$$\langle (x_2, x_3), (y_2, y_3) \rangle = x_2 y_2 + x_3 y_3$$

- $H \equiv H_1 \times H_2 = \mathbb{R}^3$

$$x = (\underline{\underline{x_1}}, \underline{x_2, x_3}) \in H_1 \times H_2 \equiv H$$

$$y = (\underset{\in H_1}{\underline{\underline{y_1}}}, \underset{\in H_2}{\underline{y_2, y_3}})$$

$$\langle x, y \rangle = \langle x_1, y_1 \rangle_1 + \langle (x_2, x_3), (y_2, y_3) \rangle_2 = x_1 y_1 + x_2 y_2 + x_3 y_3 .$$

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\langle x_1, x_1 \rangle_1 + \langle (x_2, x_3), (x_2, x_3) \rangle_2} = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

ex

$H_1 = \mathbb{R}^2 \quad \langle \cdot, \cdot \rangle_1$

$H_2 = C([0,1])$

where $\langle f, g \rangle_2 = \int_0^1 f(t) g(t) dt$

Let $x = (-2, -1)$, $y = (-1, 1) \in H_1 (= \mathbb{R}^2)$.

Let $\begin{cases} f(t) = t \ ({}^\forall t \in [0,1]) \in H_2, \\ g(t) = t^2 \ ({}^\forall t \in [0,1]) \in H_2. \end{cases}$

Define $\begin{cases} X = (x, f) \in H_1 \times H_2 = H, \\ Y = (y, g) \in H_1 \times H_2 = H. \end{cases}$

Then,

$$\begin{aligned} \langle X, Y \rangle &= \langle (x, f), (y, g) \rangle \\ &= \langle x, y \rangle_1 + \langle f, g \rangle_2 \\ &= \langle (-2, -1), (-1, 1) \rangle_1 \\ &\qquad + \int_0^1 f(t) g(t) dt \\ &= 1 + \int_0^1 t^3 dt = \frac{5}{4} \end{aligned}$$

Furthermore,

$$\langle X, X \rangle = \langle (x, f), (x, f) \rangle$$
$$= \langle x, x \rangle_1 + \langle f, f \rangle_2$$
$$= \langle (-2, -1), (-2, -1) \rangle_1 + \int_0^1 t^2 dt$$
$$= 5 + \frac{1}{3} = \frac{16}{3}$$

$$\therefore \|X\| = \|(x, f)\|$$
$$= \sqrt{\langle X, X \rangle} = \sqrt{\frac{16}{3}}$$

$$\langle Y, Y \rangle = \langle (y, g), (y, g) \rangle$$
$$= \langle y, y \rangle_1 + \langle g, g \rangle_2$$
$$= \langle (-1, 1), (-1, 1) \rangle_1 + \int_0^1 t^4 dt$$
$$= 2 + \frac{1}{5} = \frac{11}{5}$$

$$\therefore \|Y\| = \sqrt{\langle Y, Y \rangle} = \sqrt{\frac{11}{5}}$$

H_1, H_2 $\mathbb{R}$-pre-Hilbert spaces
$H = H_1 \times H_2$ product space
$\{x_n\} \subset H$ where $x_n = (u_n, v_n) \in H_1 \times H_2$.
$\Rightarrow$ Equivalent
① $\{x_n\} \subset H$: bdd
② $\{u_n\} \subset H_1$, $\{v_n\} \subset H_2$: bdd

Proof.

Note that $\|x_n\| = \sqrt{\|u_n\|_1^2 + \|v_n\|_2^2}$. —(*)

①⇒②

From ①, $\exists M \geqq 0 : \forall n \in \mathbb{N}, \|x_n\| \leqq M$. —(**)

For $n \in \mathbb{N}$,

$$\|u_n\|_1 \leqq \sqrt{\|u_n\|_1^2 + \|v_n\|_2^2} \overset{(*)}{=} \|x_n\| \overset{(**)}{\leqq} M.$$

$\therefore \exists M \geqq 0 : \forall n \in \mathbb{N}, \|u_n\|_1 \leqq M$.

This indicates that $\{u_n\} \subset H_1$ is bdd.

Similarly, $\{v_n\} \subset H_2$ is also bdd. $\lrcorner$

② ⇒ ①

From ②,

$\exists M_1 \geqq 0 : \forall n \in \mathbb{N}, \|u_n\|_1 \leqq M_1,$

$\exists M_2 \geqq 0 : \forall n \in \mathbb{N}, \|v_n\|_2 \leqq M_2.$

Define $M \equiv \sqrt{M_1^2 + M_2^2} \geqq 0.$

For $n \in \mathbb{N}$, we have that

$$\|x_n\| = \sqrt{\|u_n\|_1^2 + \|v_n\|_2^2} \quad \leftarrow (*)$$

$$\leqq \sqrt{M_1^2 + M_2^2} \equiv M.$$

$\therefore \exists M \geqq 0 : \forall n \in \mathbb{N}, \|x_n\| \leqq M.$

This shows that

$\{x_n\} \subset H$ is bdd.

//

H_1, H_2 $\mathbb{R}$-pre-Hilbert spaces

$H \equiv H_1 \times H_2$ product space

$x_n \equiv (u_n, v_n) \in H$

$x \equiv (u, v) \in H$

$\Longrightarrow$ Equivalent

① $x_n \to x$ in H.

② $\begin{cases} u_n \to u \text{ in } H_1 \\ v_n \to v \text{ in } H_2 \end{cases}$

Proof

Note that

$$\begin{aligned} \|x_n - x\| &= \|(u_n, v_n) - (u, v)\| \\ &= \|(u_n - u, v_n - v)\| \\ &= \sqrt{\|u_n - u\|_1^2 + \|v_n - v\|_2^2}. \end{aligned}$$

Therefore, ① $\Leftrightarrow$ ② holds.

//

H_1, H_2 $\mathbb{R}$-pre-Hilbert spaces

$H \equiv H_1 \times H_2$ product space

$x_n \equiv (u_n, v_n) \in H_1 \times H_2 \equiv H$

$\Rightarrow$ Equivalent

① $\{x_n\}$: Cauchy sequence in H.

② $\left.\begin{array}{l}\{u_n\} \subset H_1 \\ \{v_n\} \subset H_2\end{array}\right)$: Cauchy sequences

Th (Bolzano-Weierstrass)

$$\{x_n\} \subset \mathbb{R}^N \text{ bdd}$$
$$\Rightarrow \exists \{x_{n_i}\} \subset \{x_n\},\ x \in \mathbb{R}^N : x_{n_i} \to x$$

Proof ($N=2$)

Let $x_n = (u_n, v_n) \in \mathbb{R}^2$.

As $\{x_n\}$ is bdd,

$\{u_n\}, \{v_n\} (\subset \mathbb{R})$ are also bdd.

Thus, $\exists \{u_{n_i}\} \subset \{u_n\},\ u \in \mathbb{R} : u_{n_i} \to u$.

As $\{v_{n_i}\}$ is bdd,

$\exists \{v_{n_j}\} \subset \{v_{n_i}\},\ v \in \mathbb{R} : v_{n_j} \to v$.

Note that as $\{u_{n_j}\} \subset \{u_{n_i}\}$,

it holds that $u_{n_j} \to u$.

Hence, $\exists \{x_{n_j}\} \subset \{x_n\},\ x \equiv (u, v) \in \mathbb{R}^2 :$

$$x_{n_j} \to x.$$

//

Th

$C \subset \mathbb{R}^N \neq \emptyset$, closed, bdd

$f: C \to \mathbb{R}$ continuous

$\Rightarrow$ ① $\exists x^* \in C : f(x^*) = \sup_{x \in C} f(x)$

② $\exists x_* \in C : f(x_*) = \inf_{x \in C} f(x)$

Proof

①: Define $\alpha \equiv \sup_{x \in C} f(x) \in (-\infty, \infty]$.

Then, $\exists \{x_n\} \subset C : f(x_n) \to \alpha$. — (*)

As C is bdd, so is $\{x_n\}$.

We have that

$\exists \{x_{n_i}\} \subset \{x_n\}, x^* \in \mathbb{R}^N : x_{n_i} \to x^*$ — (**)

As $\{x_{n_i}\} \subset (\{x_n\} \subset) C$, $x_{n_i} \to x^*$, and

C is closed in $\mathbb{R}^N$, it holds that

$x^* \in C$.

$\therefore f(x^*) \in \mathbb{R}$ exists.

From (*) and (**),

$$\alpha = \lim_{n \to \infty} f(x_n) = \lim_{i \to \infty} f(x_{n_i})$$

$$= f\left(\lim_{i \to \infty} x_{n_i}\right) = f(x^*).$$

$\lrcorner$ //

Th

H_1, H_2 $\mathbb{R}$-pre-Hilbert spaces
$H \equiv H_1 \times H_2$ product space
$\Rightarrow$ Equivalent
① H : complete
② H_1, H_2 : complete

Proof

① $\Rightarrow$ ②

Let $\{u_n\} \subset H_1$ and $\{v_n\} \subset H_2$ be Cauchy sequences in H_1 and H_2, respectively.

Define $x_n = (u_n, v_n) \in H$.
Then, $\{x_n\}$ $(\subset H)$ is a Cauchy sequence.
From ①, $\exists x = (u, v) \in H_1 \times H_2 \, (=H)$:

$$x_n = (u_n, v_n) \to (u, v) = x.$$

This indicates that $\begin{cases} u_n \to u \\ v_n \to v. \end{cases}$

Thus, H_1 and H_2 are complete.

② $\Rightarrow$ ①

Let $\{x_n\} \subset H$ be a Cauchy sequence.

Let $x_n = (u_n, v_n) \in H_1 \times H_2 = H$.

Then, $\{u_n\} (\subset H_1)$ and $\{v_n\} (\subset H_2)$ be Cauchy sequences in H_1 and H_2, respectively.

From ②, $\begin{cases} \exists u \in H_1 : u_n \to u, \\ \exists v \in H_2 : v_n \to v. \end{cases}$

Define $x = (u, v) \in H$.

Then, $x_n (= (u_n, v_n)) \to x (= (u, v))$.

This implies that H is complete. //

Cor

$\boxed{\mathbb{R}^N \text{ is complete.}}$

Cor

$\boxed{\mathbb{R}^N \text{ is a } \mathbb{R}\text{-Hilbert space.}}$

$$\left.\begin{array}{l}(H_1, \langle\cdot,\cdot\rangle_1) \\ (H_2, \langle\cdot,\cdot\rangle_2)\end{array}\right\}\ \mathbb{R}\text{-pre-Hilbert space}$$

$H \equiv H_1 \times H_2$ product space

where

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle_1 + \langle x_2, y_2 \rangle_2$$

Then,

$$\boxed{\begin{array}{c}(x_1, x_2) \perp (y_1, y_2) \\ \text{in } H\end{array}} \begin{array}{c}\Longleftarrow \\ \not\Longrightarrow\end{array} \boxed{\begin{array}{l}x_1 \perp y_1 \text{ in } H_1 \\ x_2 \perp y_2 \text{ in } H_2\end{array}}$$

<u>ex</u>

$H_1 = H_2 = \mathbb{R}$

Then, $H \equiv H_1 \times H_2 = \mathbb{R}^2$

Let $\left\{\begin{array}{l}(x_1, x_2) = (1, -1) \in H \\ (y_1, y_2) = (1, 1) \in H.\end{array}\right.$

Then, we have $(x_1, x_2) \perp (y_1, y_2)$ in $\mathbb{R}^2$.

However, $\left\{\begin{array}{l}x_1 \not\perp y_1 \\ x_2 \not\perp y_2\end{array}\right.$.

Angle, orthogonality, and product spaces

1. 実プレ・ヒルベルト空間において，角度がどのように定義されるか説明せよ．また，複素プレ・ヒルベルト空間では角度を定義するのにどのような支障があるか？

2. 実プレ・ヒルベルト空間Hにおいて，$x \perp y \iff \frac{x}{\|x\|} \perp \frac{y}{\|y\|}$はいえるか？

3. 集合$C([-1,1])=\{f : [-1,1] \to \mathbb{R} \mid f \text{ is continuous.}\}$において，$\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)dx$と定義した実プレ・ヒルベルト空間を考える．ここで直交するふたつの要素を挙げよ．

4. ふたつの複素ベクトル空間V_1, V_2とその直積集合$V = V_1 \times V_2$を考える．ここに（レジュメの本文中で定義した）自然な方法で和とスカラー倍を導入すると，Vは複素ベクトル空間になる．これを示せ．

5. ふたつの複素プレ・ヒルベルト空間$(H_1, \langle\cdot,\cdot\rangle_1)$, $(H_2, \langle\cdot,\cdot\rangle_2)$とその直積集合$H = H_1 \times H_2$を考える．ここに標準的な意味で和とスカラー倍を導入すると，Hは複素ベクトル空間になる（この問題では，この点は認めればよい）．さらに，$x = (x_1, x_2)$, $y = (y_1, y_2) \in H$に対してそれらの内積を

$$\langle x,y\rangle = \langle x_1,y_1\rangle_1 + \langle x_2,y_2\rangle_2$$

と定めると，$(H, \langle\cdot,\cdot\rangle)$は複素内積空間になる．このことを証明せよ．ただし，ここで$x_1, y_1 \in H_1$, $x_2, y_2 \in H_2$である．

6. ふたつの実プレ・ヒルベルト空間$(H_1, \langle\cdot,\cdot\rangle_1)$, $(H_2, \langle\cdot,\cdot\rangle_2)$として，$H_1 = \mathbb{R}^2$, $H_2 = C([0,1])$とする．ここで，H_2における内積は，

$$\langle f,g\rangle_2 = \int_0^1 f(t)g(t)dt$$

と定める．問題5で導入した形で直積空間$H = H_1 \times H_2$を考える．H_1とH_2のふたつずつの要素

$$x = (1,-1) \in H_1 = \mathbb{R}^2,$$
$$y = (-2,-1) \in H_1 = \mathbb{R}^2,$$
$$f(t) = t \in H_2 = C([0,1]),$$
$$g(t) = -t \in H_2 = C([0,1])$$

に対して，$H(= H_1 \times H_2)$の要素を

$$X = (x,f) \in H, \quad Y = (y,g) \in H$$

とする．このとき，$\langle X, Y\rangle$, $\|X\|$, $\|Y\|$を求めよ．

7. ふたつの実プレ・ヒルベルト空間$(H_1, \langle\cdot,\cdot\rangle_1)$, $(H_2, \langle\cdot,\cdot\rangle_2)$に対して，$H = H_1 \times H_2$を問題5で導入した直積空間とする．$\{x_n = (u_n, v_n)\}$を$H$内の点列とする．ここで，$\{u_n\} \subset H_1$, $\{v_n\} \subset H_2$である．このとき，以下の2条件は同値である．これを示せ．

(1) $\{x_n\}$はHの有界点列である．

(2) $\{u_n\}, \{v_n\}$は，それぞれH_1, H_2の有界点列である．

8. 問題7と同じ設定を考える. また, $x_n = (u_n, v_n)$, $x = (u, v) \in H$とする. このとき, 以下の2条件は同値であることを確認せよ.
(1) $x_n \to x$,
(2) $u_n \to u$ and $v_n \to v$.

9. 問題7と同じ設定を考え, $x_n = (u_n, v_n) \in H$とする. このとき, 以下の2条件は同値であることを確認せよ.
(1) $\{x_n\}$はHのコーシー列である.
(2) $\{u_n\}, \{v_n\}$は, それぞれH_1, H_2のコーシー列である.

10. (Bolzano–Weierstrassの定理) $\{x_n\}$は$\mathbb{R}^2$の有界点列とする. このとき, $\{x_n\}$が収束部分列を含むことを示せ.

11. (最大値・最小値の定理) Cを$\mathbb{R}^N$の空ではない有界閉集合, fはC上で定義された実数値連続関数とする. このとき, fはC上で最大値と最小値をとることを示せ.

12. 問題7と同じ設定で, Hが完備であることと, H_1とH_2の両方が完備であること同値である. このことを示せ.

13. $\mathbb{R}^N$が完備であることを確認せよ.

解答.
6. $\langle X, Y\rangle = -4/3$, $\|X\| = \sqrt{5/2}$, $\|Y\| = \sqrt{11/2}$.