Orthogonal direct sum decomposition

Def.

H $\mathbb{R}$-pre-Hilbert space

$A \subset H$, $\neq \phi$

$A^{\perp} = \{ x \in H \mid {}^{\forall} y \in A, \langle x, y \rangle = 0 \}$

orthogonal complement

直交補空間

Remark.

$A^{\perp} \neq \phi$

($\because$) $0 \in A^{\perp}$. //

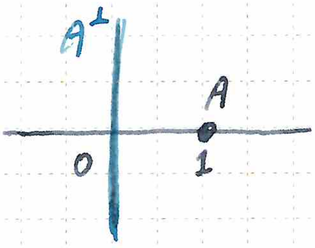
ex

$H = \mathbb{R}^2$

$A = \{(1,0)\}$

Then,

$$A^{\perp} = \{(x,y) \in \mathbb{R}^2 \mid \forall (u,v) \in A, \langle (x,y),(u,v) \rangle = 0\}$$
$$= \{(x,y) \in \mathbb{R}^2 \mid x = 0\}$$

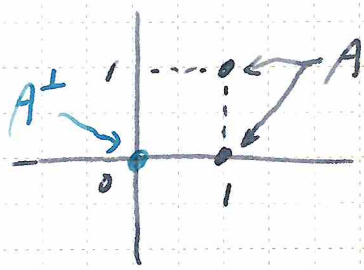


ex

$H = \mathbb{R}^2$

$A = \{(1,0),(1,1)\}$

Then, $A^{\perp} = \{(x,y) \in \mathbb{R}^2 \mid x = 0,\ x + y = 0\}$
$= \{(0,0)\}$



$$\boxed{\begin{array}{l} H \ \ \mathbb{R}\text{-pre-Hilbert space} \\ \Longrightarrow \{0\}^{\perp} = H \end{array}}$$

Proof

($\subset$) OK.

($\supset$) Let $x \in H$.

We prove that $x \in \{0\}^{\perp}$.

i.e. $\forall y \in \{0\}, \langle x, y \rangle = 0$

i.e. $\langle x, 0 \rangle = 0$.

OK. //

H $\mathbb{R}$-pre-Hilbert space
$\Rightarrow H^{\perp} = \{0\}$

<u>Proof</u>

($\subset$) Let $x \in H^{\perp}$.

i.e. $\forall y \in H, \langle x, y \rangle = 0$ — (*)

We prove that $\underline{x = 0}$.

Letting $y = x$ in (*), we have

$\langle x, x \rangle = 0$. $\therefore x = 0$. ⌟

($\supset$) We show that $\underline{0 \in H^{\perp}}$.

i.e. $\forall y \in H, \langle 0, y \rangle = 0$.

OK.

//

H $\mathbb{R}$-pre-Hilbert space
$A, B \subset H, \neq \emptyset$
$A \subset B$
$\Rightarrow A^{\perp} \supset B^{\perp}$

Proof.

Let $x \in B^{\perp}$.

i.e. $\forall y \in B, \langle x, y \rangle = 0.$ — (*)

We prove that $\underline{x \in A^{\perp}}$.

i.e. $\forall y \in A, \langle x, y \rangle = 0.$

Let $y \in A$.

As $A \subset B$, it holds that $y \in B$.

Using (*) results in

$$\langle x, y \rangle = 0 \quad (\forall y \in A).$$

This indicates that $x \in A^{\perp}$.

//

H $\mathbb{R}$-preHilbert space
$A \subset H$, $\neq \phi$
$\Rightarrow A^\perp (\subset H)$ is a closed subspace.

Proof

$A^\perp (\subset H)$: subspace

Let $x, y \in A^\perp$ and let $a, b \in \mathbb{R}$.
We prove that $ax + by \in A^\perp$.
i.e. ${}^\forall z \in A$, $\langle ax+by, z \rangle = 0$.

As $x, y \in A^\perp$, we have

内積の
線型性

$\langle ax+by, z \rangle$
$= a\langle x, z\rangle + b\langle y, z\rangle = 0$.

$A^\perp (\subset H)$: closed.

Let $(x_n) \subset A^\perp$: $x_n \to x \in H$.
i.e. ${}^\forall n \in \mathbb{N}$, $z \in A$, $\langle x_n, z \rangle = 0$
We prove that $x \in A^\perp$.
i.e. ${}^\forall z \in A$, $\langle x, z \rangle = 0$.

It follows that

内積の
連続性

$\langle x, z \rangle = \langle \lim x_n, z \rangle$
$= \lim \langle x_n, z \rangle = 0$. //

H $\mathbb{R}$-pre-Hilbert space
$A \subset H$, $\neq \emptyset$
$\Rightarrow A \cap A^{\perp} \subset \{0\}$

<u>Proof</u>

Let $x \in A \cap A^{\perp}$.

As $x \in A^{\perp}$, it holds that

$$\langle x, y \rangle = 0 \quad \forall y \in A.$$

As $x \in A$, we obtain

$$\langle x, x \rangle = 0.$$

$$\therefore x = 0.$$

//

<u>Cor</u>

H $\mathbb{R}$-pre-Hilbert space
$M \subset H$ subspace
$\Rightarrow M \cap M^{\perp} = \{0\}$

Def:

V vector space

$M_1, M_2 \, (\subset V)$ subspaces

$V = M_1 \oplus M_2$ <u>direct sum</u>

$\Leftrightarrow {}^\forall x \in V, \ \exists^1 (y,z) \in M_1 \times M_2 :$

$$x = y + z$$

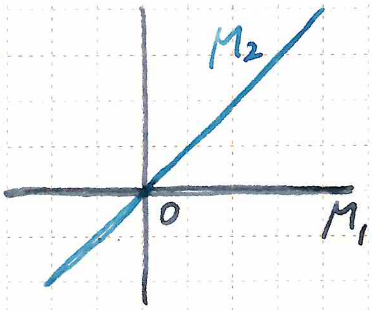
直和

ex

$V = \mathbb{R}^2$

$M_1 = \{(x,0) \mid x \in \mathbb{R}\}$

$M_2 = \{(x,x) \mid x \in \mathbb{R}\}$



Then,

$\mathbb{R}^2 = M_1 \oplus M_2$.

Proof

Let $(x,y) \in \mathbb{R}^2$.

Suppose that $(x,y) = (u,0) + (v,v)$.

i.e $\begin{cases} x = u+v \\ y = v \end{cases}$

We have $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$.

As $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is invertible,

$\exists! \, (u,0) \in M_1, \, (v,v) \in M_2 :$

$(x,y) = (u,0) + (v,v)$.

Indeed, $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$\therefore \begin{cases} u = x - y \\ v = y \end{cases}$.

//

ex

$$\boxed{\begin{aligned} &V=\mathbb{R}^2 \\ &A=\{(x,0)\mid x\in\mathbb{R}\} \\ &B=\mathbb{R}^2 \\ &\Rightarrow \mathbb{R}^2 \neq A\oplus B \end{aligned}}$$

(∵) We show that

$$\boxed{\begin{aligned} &\exists (x,y)\in\mathbb{R}^2 \\ &\qquad (a_1,0),(a_2,0)\in A \\ &\qquad (u_1,v_1),(u_2,v_2)\in B=\mathbb{R}^2 \\ &\text{s.t. } (x,y)=(a_1,0)+(u_1,v_1) \\ &\qquad\qquad\quad =(a_2,0)+(u_2,v_2) \end{aligned}}$$

Let $(1,1)\in\mathbb{R}^2$.

Then, $(1,1)=(0,0)+(1,1)$

$$= \underset{\substack{\cap \\ A}}{(1,0)} + \underset{\substack{\cap \\ B}}{(0,1)}$$

$\therefore \mathbb{R}^2 \neq A\oplus B$. //

Th

H $\mathbb{R}$-pre-Hilbert space

$M \subset H$ subspace

$x \in H$, $\bar{x} \in M$

$\Rightarrow$ Equivalent

① $\|x - \bar{x}\| \leqq \|x - y\|$ $\forall y \in M$,

② $\langle x - \bar{x}, \bar{x} - y \rangle \geqq 0$ $\forall y \in M$,

★ ③ $\langle x - \bar{x}, y \rangle = 0$ $\forall y \in M$,

★ ④ $\|x - \bar{x}\|^2 + \|\bar{x} - y\|^2 = \|x - y\|^2$ $\forall y \in M$,

⑤ $\|x - \bar{x}\|^2 + \|\bar{x} - y\|^2 \leqq \|x - y\|^2$ $\forall y \in M$.

Proof.

③ $\Rightarrow$ ④

Let $y \in M$.

As $\bar{x}, y \in M$ and $M (\subset H)$ is a subspace, it follows from ③ that

$$2\langle x - \bar{x}, \bar{x} - y \rangle = 0.$$

$$\therefore \|x - y\|^2 + \|\bar{x} - \bar{x}\|^2 - \|x - \bar{x}\|^2 - \|\bar{x} - y\|^2 = 0.$$

$$\therefore \|x - y\|^2 = \|x - \bar{x}\|^2 + \|\bar{x} - y\|^2 \quad (\forall y \in M).$$

$\lrcorner$

$$\left.\begin{array}{l} \underline{④ \Rightarrow ⑤} \\ \underline{⑤ \Rightarrow ①} \\ \underline{① \Rightarrow ②} \end{array}\right\} \text{OK.}$$

$\underline{② \Rightarrow ③}$

Let $z \in M$.

We prove that $\underline{\langle x - \bar{x}, z \rangle = 0}$.

As $\bar{x}, z \in M$ and $M (\subset H)$ is a subspace, we have that $\bar{x} - z, \bar{x} + z \in M$.

Setting $y = \bar{x} - z \in M$ in ②, we have

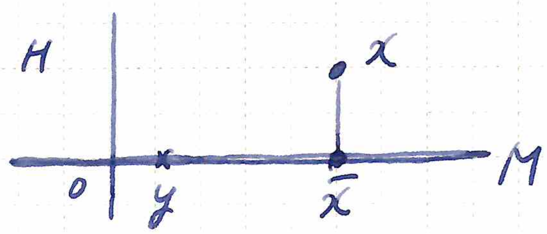
$$\langle x - \bar{x}, \bar{x} - (\bar{x} - z) \rangle \geqq 0.$$

$$\therefore \langle x - \bar{x}, z \rangle \geqq 0.$$

Similarly, setting $y = \bar{x} + z \in M$ in ②, we obtain $\langle x - \bar{x}, z \rangle \leqq 0$.

Consequently, $\langle x - \bar{x}, z \rangle = 0 \quad \forall z \in M$. //

$$\boxed{※: \quad ③ \Leftrightarrow x - \bar{x} \in M^{\perp}}$$



Th

H $\mathbb{R}$-Hilbert space
$M \subset H$ closed subspace
$\Rightarrow H = M \oplus M^{\perp}$

i.e. $\forall x \in H, \exists^1 (y, z) \in M \times M^{\perp} : x = y + z$

Proof.

Let $x \in H$.

<Existence>

From the nearest point theorem,

$\exists^1 y \in M : \|x - y\| = d(x, M)$.

Define $z = x - y \in H$.

Then, $z \in M^{\perp}$.

$\therefore \forall x \in H, \exists (y, z) \in M \times M^{\perp} : x = y + z$. ⌟

<Uniqueness>

Let $x = y_1 + z_1$
$= y_2 + z_2$

where $y_1, y_2 \in M$ and $z_1, z_2 \in M^{\perp}$.

We have $y_1 - y_2 = z_2 - z_1 \in M \cap M^{\perp} = \{0\}$.

$\therefore y_1 = y_2$ and $z_1 = z_2$. //

Cor

H $\mathbb{R}$-Hilbert space

$f \in H^* \equiv \{ g: H \to \mathbb{R} \mid g: \text{linear, continuous.} \}$

$M \equiv \operatorname{Ker} f$

$\Rightarrow H = M \oplus M^{\perp}$

H $\mathbb{R}$-pre-Hilbert space
$A \subset H$, $\neq \emptyset$
$\Rightarrow A \subset A^{\perp\perp}$

Proof

Let $x \in A$.

We prove that $x \in A^{\perp\perp}$.

i.e. $\forall y \in A^{\perp}$, $\langle x, y \rangle = 0$.

As $x \in A$, OK. //

ex

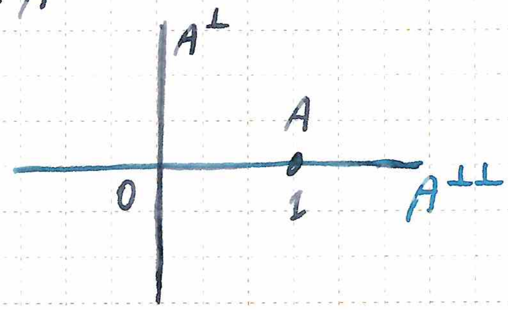
$H = \mathbb{R}^2$

$A = \{(1,0)\}$

Then, $A^{\perp} = Y$-axis

$A^{\perp\perp} = X$-axis.

$\therefore A \subset A^{\perp\perp}$



Th

H $\mathbb{R}$-Hilbert space
M ($\subset$ H) closed subspace
$\Rightarrow M = M^{\perp\perp}$

Proof

It is sufficient to prove that
$M \supset M^{\perp\perp}$.

Let $x \in M^{\perp\perp}$ ($\subset H$).
We prove that $x \in M$.
For $x \in (M^{\perp\perp} \subset) H$,
$$\exists 1 (y, z) \in M \times M^{\perp} : x = y + z. \quad -(*)$$
To prove that $x \in M$, we demonstrate that $z = 0$.
From (*), $z = x - y$.
As $x \in M^{\perp\perp}$ and $y \in M \subset M^{\perp\perp}$,
we obtain $z = x - y \in M^{\perp\perp}$.
Hence, $z \in M^{\perp} \cap M^{\perp\perp} = \{0\}$.
$\therefore z = 0$. //

H $\mathbb{R}$-Hilbert space
$M_1, M_2 (\subset H)$ closed subspaces
Then, $M_1 = M_2 \Leftrightarrow M_1^\perp = M_2^\perp$

<u>Proof</u>

($\Rightarrow$) OK.

($\Leftarrow$) Assume that $M_1^\perp = M_2^\perp$.

Then, $M_1^{\perp\perp} = M_2^{\perp\perp}$

As M_1, M_2 are closed subspaces,

$M_1^{\perp\perp} = M_1$ and $M_2^{\perp\perp} = M_2$.

$\therefore M_1 = M_1^{\perp\perp} = M_2^{\perp\perp} = M_2$.

$\therefore M_1 = M_2$. //

Orthogonal direct sum decompositions

1. Aを実プレ・ヒルベルト空間Hの空でない部分集合とする. Aの直交補空間$A^{\perp}$の定義を述べ, 例を挙げて説明せよ.

2. 実プレ・ヒルベルト空間Hにおいて, $H^{\perp} = \{0\}$ と $\{0\}^{\perp} = H$ を証明せよ.

3. $\mathbb{R}^2$において, X軸$= \{(x, 0) \mid x \in \mathbb{R}\}$の直交補空間は$Y$軸であることを示せ.

4. A, Bを実プレ・ヒルベルト空間Hの空でない部分集合で$A \subset B$とする. このとき, $A^{\perp} \supset B^{\perp}$を示せ.

5. Aを実プレ・ヒルベルト空間Hの空でない部分集合とする. このとき, $A^{\perp}$はHの閉部分空間になることを示せ.

6. Aを実プレ・ヒルベルト空間Hの空でない部分集合とする. このとき, $A \cap A^{\perp} \subset \{0\}$となることを示せ. また, Aが部分空間の場合は, 逆の包含関係もいえる. なぜか?

7. ベクトル空間のふたつの部分空間について, それらの直和の定義を述べよ.

8. $\mathbb{R}^2$のふたつの部分空間

$$M_1 = \{(x,y) \mid y = x\}, \quad M_2 = \{(x,y) \mid y = -x\}$$

について, $\mathbb{R}^2 = M_1 \oplus M_2$となることを証明せよ.

9. $\mathbb{R}^2$のふたつの部分空間

$$M_1 = \{(x,y) \mid x = y\}, \quad M_2 = \mathbb{R}^2$$

について, $\mathbb{R}^2 \neq M_1 \oplus M_2$である. 実際に, $\mathbb{R}^2$の要素について, 二通りの方法でM_1とM_2の要素の和として表してみよ.

10. Mを実プレ・ヒルベルト空間Hの部分空間とする. また, $x \in H$, $\bar{x} \in M$とする. このとき, 次の5条件が同値であることを証明せよ.

(1) $\|x - \bar{x}\| \leq \|x - y\|$ for all $y \in M$,

(2) $\langle x - \bar{x}, \bar{x} - y\rangle \geq 0$ for all $y \in M$,

(3) $\langle x - \bar{x}, y\rangle = 0$ for all $y \in M$,

(4) $\|x - \bar{x}\|^2 + \|\bar{x} - y\|^2 = \|x - y\|^2$ for all $y \in M$,

(5) $\|x - \bar{x}\|^2 + \|\bar{x} - y\|^2 \leq \|x - y\|^2$ for all $y \in M$.

11. Mを実ヒルベルト空間Hの閉部分空間とする. このとき, $H = M \oplus M^{\perp}$となることを証明せよ. また, 証明の中でHの完備性が効いている箇所を指摘せよ.

12. Mを実ヒルベルト空間Hの閉部分空間とする. このとき, $M = M^{\perp\perp}$となることを証明せよ. また, $M \subset M^{\perp\perp}$については, Mについて特に仮定は必要ないことを確認せよ.

13. 実ヒルベルト空間Hのふたつの閉部分空間M_1, M_2について, $M_1 = M_2 \Leftrightarrow M_1^{\perp} = M_2^{\perp}$を証明せよ.