Variational inequality problems

Review

Def.

X, Y MSs

$A : X \to Y$ K-Lipschitz

$\Leftrightarrow \exists K \geqq 0 : \forall x, y \in X,$

$$d(Ax, Ay) \leqq K d(x, y)$$

X, Y, Z MSs

$A : X \to Y$ K-Lipschitz

$B : Y \to Z$ L-Lipschitz

$\Rightarrow B \circ A : X \to Z$ KL-Lipschitz

$C \subset \mathbb{R} \neq \emptyset$, convex

$A : C \to \mathbb{R}$ differentiable

$K \geqq 0$

$\Rightarrow$ Equivalent

① A: K-Lipschitz

② $\forall x \in C, \ |A'(x)| \leqq K$

Def:

H $\mathbb{R}$-Hilbert space

$C \subset H$, $\neq \emptyset$

$A: C \to H$ monotone

$\Leftrightarrow$ $\forall x, y \in C$, $\langle x-y, Ax-Ay \rangle \geqq 0$

$C \subset \mathbb{R}$

$A: C \to \mathbb{R}$

$\Rightarrow$ Equivalent

① A: monotone

② A: monotone increasing

i.e. $x \leqq y \Rightarrow Ax \leqq Ay$

$M \subset H$ subspace

$A: M \to H$ linear

$\Rightarrow$ Equivalent

① A: monotone

② $\forall x \in M$, $\langle Ax, x \rangle \geqq 0$

Def

$C \subset H, \neq \emptyset$

$A : C \to H$ $\underline{\eta\text{-strongly monotone}}$

$(\eta\text{-SM})$

$\Leftrightarrow \exists \eta > 0 : \forall x, y \in C,$

$$\eta \|x-y\|^2 \leq \langle x-y, Ax-Ay \rangle$$

$C \subset H, \neq \emptyset$

$A : C \to H$ $\eta\text{-SM}$

$\Rightarrow A :$ strongly monotone

i.e $x, y \in C,$

$$x \neq y \Rightarrow \langle x-y, Ax-Ay \rangle > 0$$

$C \subset \mathbb{R}$, $\neq \emptyset$, convex

$\eta > 0$

$A : C \to \mathbb{R}$ differentiable

$\Rightarrow$ Equivalent

① A : η-SM

② $\forall x \in C$, $\eta \leq A'(x)$

Proof

①$\Rightarrow$②

Let $x \in C$ and let $y \in C : y \neq x$.

From ①, $\eta(x-y)^2 \leq (x-y)(Ax - Ay)$.

As $x \neq y$, $\eta \leq \dfrac{Ax - Ay}{x-y}$.

As $y \to x$, we obtain $\eta \leq A'(x)$ $\forall x \in C$.

②$\Rightarrow$①

Let $x, y \in C$.

W.l.g., assume that $x < y$.

Using ② and the mean value theorem, we have

$\exists z \in (x, y)$: $\eta \leq A'(z) = \dfrac{Ax - Ay}{x-y}$.

Multiplying $(x-y)^2 (>0)$, we obtain

$\eta(x-y)^2 \leq (x-y)(Ax - Ay)$.

Similarly, this holds for the case $x > y$.

This completes the proof.

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$C \subset H, \neq \emptyset$

$A : C \to H$

$VI(C, A) = \{x \in C \mid \forall y \in C, \langle y - x, Ax \rangle \geq 0\}$

variational inequality

◎ Optimization problem

$$\boxed{\begin{array}{l} A : H \to H \\ \Rightarrow VI(H, A) = A^{-1}0 \end{array}}$$

Proof

Note that

$x \in VI(H, A)$

$\Leftrightarrow \forall y \in H, \langle y - x, Ax \rangle \geqq 0.$ — (*)

($\supset$) OK.

($\subset$) Let $x \in VI(H, A)$.

Setting $y = x - Ax \in H$ in (*), we have

$\langle (x - Ax) - x, Ax \rangle \geqq 0.$

$\therefore \langle Ax, Ax \rangle \leqq 0.$

$\therefore Ax = 0.$

$\therefore x \in A^{-1}0.$ //

$f \in C^1([a,b], \mathbb{R})$
$x^* \in [a,b]$
$\Rightarrow$ Equivalent
① (i) $x^* \in (a,b) \Rightarrow f'(x^*) = 0$
(ii) $x^* = a \Rightarrow f'(x^*) \geqq 0$
(iii) $x^* = b \Rightarrow f'(x^*) \leqq 0$
② $\forall y \in [a,b],\ f'(x^*) \cdot (y - x^*) \geqq 0$

Proof

① $\Rightarrow$ ②

Let $y \in [a,b]$.
We show that $f'(x^*) \cdot (y - x^*) \geqq 0$.

(i) $x^* \in (a,b)$ OK.

(ii) $x^* = a$
From ①, $f'(x^*) \geqq 0$.
As $y - x^* = y - a \geqq 0$, OK.

(iii) $x^* = b$
From ①, $f'(x^*) \leqq 0$.
As $y - x^* = y - b \leqq 0$, OK. ┘

② $\Rightarrow$ ①

(i) $x^* \in (a, b)$

We prove that $f'(x^*) = 0$.

Let $y \in [a, b]$.

From ②,

- $a \leqq y < x^* < b \Rightarrow f'(x^*) \leqq 0$.
  $(y - x^* < 0)$
- $a < x^* < y \leqq b \Rightarrow f'(x^*) \geqq 0$.
  $(y - x^* > 0)$

Therefore, $f'(x^*) = 0$ must hold. ⌟

(ii) $x^* = a$

In this case,

$\forall y \in [a, b],\ y - x^* \geqq 0$.

From ②, $f'(x^*) \geqq 0$. ⌟

(iii) $x^* = b$

In this case,

$\forall y \in [a, b],\ y - x^* \leqq 0$.

From ②, $f'(x^*) \leqq 0$.

//

ex

$H = \mathbb{R}$

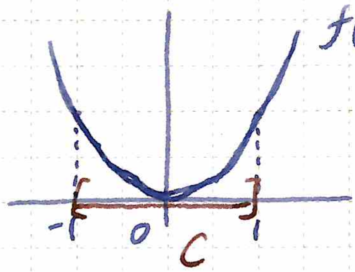
$Ax = 2x$ monotone

$\left(\begin{array}{c} f(x) = x^2 \\ \text{convex} \\ Ax = f'(x) = 2x \end{array} \right)$

- $C = [-1, 1]$

Then,

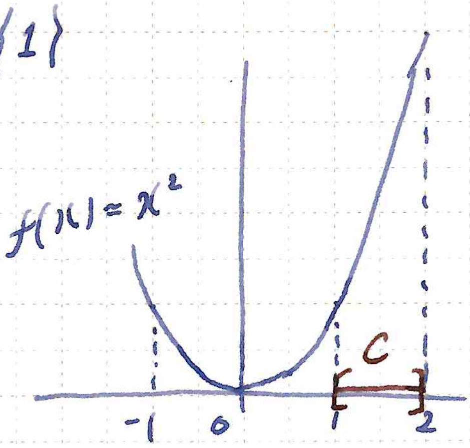
$$VI(C, A) = \{ x \in [-1,1] \mid \forall y \in [-1,1], \langle y - x, 2x \rangle \geq 0 \}$$
$$= \{ x \in [-1,1] \mid \forall y \in [-1,1], x(y-x) \geq 0 \}$$
$$= \{0\}$$



- $C = [1, 2]$

Then,

$$VI(C, A) = \{ x \in [1,2] \mid \forall y \in [1,2], x(y-x) \geq 0 \}$$
$$= \{ x \in [1,2] \mid \forall y \in [1,2], y - x \geq 0 \}$$
$$= \{1\}$$



- $C = \{-1, 1\}$

Then,

$$VI(C, A) = \{x \in \{-1, 1\} \mid \forall y \in \{-1, 1\},\ x(y-x) \geq 0\}$$
$$= \emptyset$$

<u>ex</u>

$H = \mathbb{R}$

$C = [-1, 1]$

$Ax = x^2$ not monotone

$\left(f(x) = \frac{1}{3}x^3 \right.$ not convex

$\left. A = f' \right)$

In this case,

$$VI(C, A) = \{x \in [-1, 1] \mid \forall y \in [-1, 1],\ x^2(y-x) \geq 0\}$$
$$= \{-1, 0\}$$

※ $VI(C, A)$ は無条件で最適化問題の解に結びつくわけではない。

@ Fixed point problem

$$
\boxed{
\begin{array}{l}
C \subset H,\ \neq \emptyset \\
T : C \to C \\
\Rightarrow F(T) = VI(C, I-T)
\end{array}
}
$$

Proof

Note that

$x \in VI(C, I-T)$

$\iff \forall y \in C,\ \langle y-x, x-Tx \rangle \geq 0.$ —(*)

($\subset$) OK.

($\supset$) Let $x \in VI(C, I-T)$.

As $T : C \to C$, $Tx \in C$.

Setting $y = Tx \in C$ in (*), we have

$\langle Tx - x, x - Tx \rangle \geq 0.$

$\therefore\ Tx - x = 0$

$\therefore\ x = Tx.$

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$C \subset H \neq \emptyset$, closed, convex

$P_C : H \to C$ MP

$A : C \to H$

$\mu > 0$

$\Rightarrow VI(C, A) = F(P_C(I - \mu A))$

Proof.

It follows that

$x \in F(P_C(I - \mu A))$

$\Leftrightarrow x = P_C(I - \mu A)(x)$

$\Leftrightarrow x = P_C(x - \mu A x) \in C$

$\Leftrightarrow \langle (x - \mu A x) - x, x - y \rangle \geq 0 \quad \forall y \in C$

$\Leftrightarrow \langle -\mu A x, x - y \rangle \geq 0 \quad \forall y \in C$

$\Leftrightarrow \langle A x, x - y \rangle \leq 0 \quad \forall y \in C$ $\mu > 0$

$\Leftrightarrow \langle y - x, A x \rangle \geq 0 \quad \forall y \in C$

$\Leftrightarrow x \in VI(C, A)$.

//

$C \subset H, \neq \emptyset$

$A : C \to H$ η-SM, K-Lipschitz

where $\eta \leq K$

$\mu \in \left(0, \dfrac{2\eta}{K^2}\right)$

$\Rightarrow I - \mu A$: contraction

Proof

Let $x, y \in C$.

It holds that

$$\|(I-\mu A)x - (I-\mu A)y\|^2$$

$$= \|x - y - \mu(Ax - Ay)\|^2$$

$$= \|x-y\|^2 - 2\mu \underbrace{\langle x-y, Ax-Ay \rangle}_{\geq \eta \|x-y\|^2} + \mu^2 \underbrace{\|Ax - Ay\|^2}_{\leq K^2 \|x-y\|^2}$$

$$\leq \{1 - 2\mu\eta + \mu^2 K^2\} \|x-y\|^2$$

$$= \{1 - \underbrace{\mu}_{>0} \underbrace{(2\eta - \mu K^2)}_{>0}\} \|x-y\|^2.$$

//

<u>check</u>

$$1-\mu\left(2\eta-\mu K^2\right) \geqq 0$$

i.e. $2\eta\mu - \mu^2 K^2 \leqq 1$

i.e. $-K^2\mu^2 + 2\eta\mu - 1 \leqq 0$

It holds that

$$D/4 = \eta^2 - K^2 \leqq 0$$

As $\eta \leqq K$, it is satisfied. //

Th

H $\mathbb{R}$-Hilbert space

$C \subset H \neq \emptyset$, closed, convex

$P_C : H \to C$ MP

$A : C \to H$ η-SM, K-Lipschitz

where $0 < \eta \leq K$.

$\mu \in \left(0, \dfrac{2\eta}{K^2}\right)$

$\Rightarrow$ (1) $\exists^1 x^* \in VI(C, A)$

(2) $\begin{cases} x_1 \in C \text{ given} \\ x_{n+1} = P_C(I - \mu A)x_n \ (\forall n \in \mathbb{N}) \end{cases}$

$\Rightarrow x_n \to x^*$

Proof.

As $0 < \mu < \dfrac{2\eta}{K^2}$, $I - \mu A$ is a contraction mapping.

As P_C is NE (i.e. 1-Lipschitz mapping), it holds that

$$P_C(I - \mu A) : C \to C$$

is a contraction mapping.

As H is complete and $C(\subset H)$ is closed, C is also complete.
Consequently,

$\exists^1 x^* \in F(P_C(I-\mu A)) = VI(C,A)$:

$\forall x_1 \in C,\ x_{n+1} = P_C(I-\mu A)x_n,$

$\Rightarrow x_n \to x^*.$ //

<the projected gradient method>

cor

$A : H \to H$ η-SM, K-Lipschitz
where $0 < \eta \leqq K$
$\mu \in \left(0, \frac{2\eta}{K^2}\right)$
$\Rightarrow \exists^1 x^* \in VI(H,A) = A^{-1}0$

$$\begin{cases} x_1 \in C \text{ given} \\ x_{n+1} = x_n - \mu A x_n \end{cases}$$

$\Rightarrow x_n \to x^*$

ex

$H = \mathbb{R}$

$A : \mathbb{R} \to \mathbb{R}$ defined by

$$Ax = 2x \quad (x \in \mathbb{R})$$

Then, A: 2-SM and 2-Lipschitz

Thus, $\frac{2\eta}{K^2} = 1$

It holds that

- $VI(H, A) = A^{-1}0 = \{0\}$
- $F(I - \mu A) = \{0\}$ for $\mu \in (0,1)$.

- $\mu = 1/3$

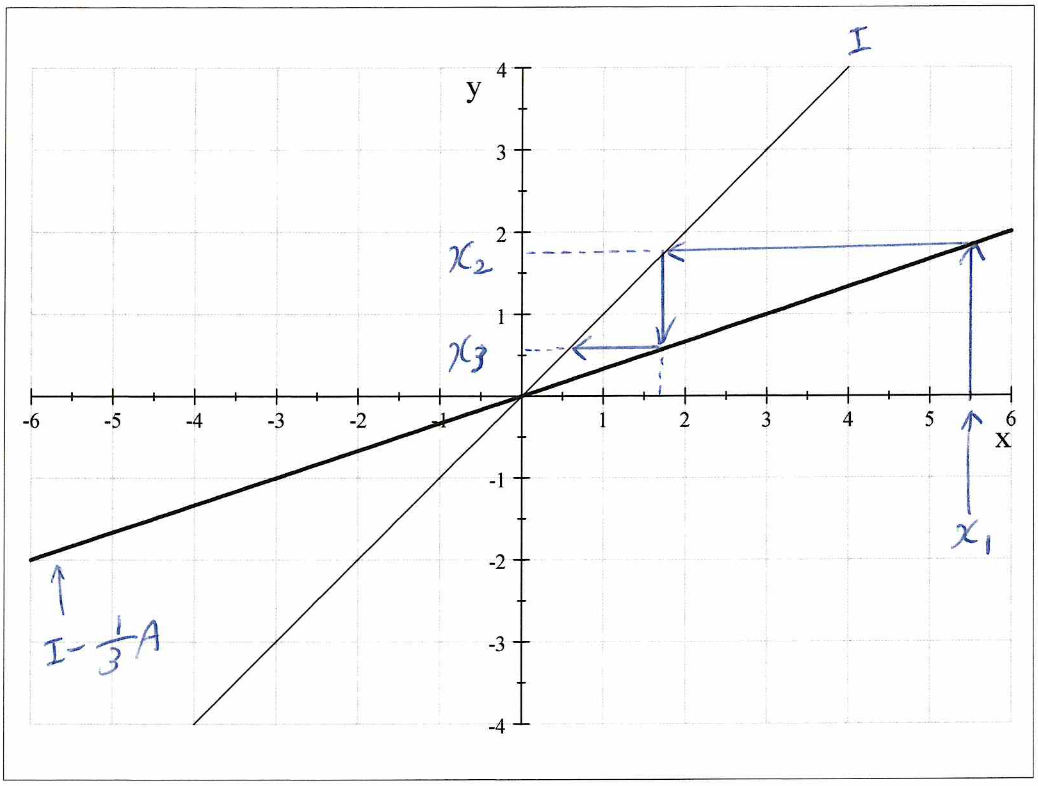
  Then, $(I - \mu A)x = x - \frac{2}{3}x = \frac{1}{3}x$

- $\mu = 1/2$

  Then, $(I - \mu A)x = 0$

- $\mu = 2/3$

  Then, $(I - \mu A)x = x - \frac{4}{3}x = -\frac{1}{3}x$



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$I - \frac{1}{3}A$

Monotone operators and VIP

1. 実ヒルベルト空間Hの非空部分集合C上で定義された作用素Aがη–strongly monotone ($\eta > 0$) であることの定義を確認せよ．また，$H = \mathbb{R}$でAが微分可能な場合には，$A : C \to \mathbb{R}$がη–strongly monotoneであることと，条件

$$\eta \leq A'(x) \quad \text{for all } x \in C$$

が同値になることを証明せよ．

2. $A : H \to H$のとき，変分不等式問題の解集合$VI(C,A)$とAの零点集合$A^{-1}0$が一致することを証明せよ．

3. 関数$f : [a,b] \to \mathbb{R}$を微分可能とし，$x^* \in [a,b]$とする．ここで，$[a,b]$は$\mathbb{R}$における区間である．このとき，x^*が変分不等式問題$VI([a,b],f')$の解であること$x^* \in VI([a,b],f')$は，次の条件で特徴付けられることを示せ．

$$\begin{array}{lcl} \text{(i)} & x^* \in (a,b) & \Rightarrow\ f'(x^*) = 0 \\ \text{(ii)} & x^* = a & \Rightarrow\ f'(x^*) \geq 0 \\ \text{(iii)} & x^* = b & \Rightarrow\ f'(x^*) \leq 0 \end{array}$$

4. 指数関数$Ax = e^x$を$C = [0,\infty)$上に制限して考える．このとき，$VI(C,A)$はどうなるか？

5. 実ヒルベルト空間Hの非空部分集合C上で定義された写像$T : C \to C$について，

$$F(T) = VI(C,\ I - T)$$

を示せ．ただし，$F(T)$はTの不動点集合であり，IはC上の恒等写像である．

6. 実ヒルベルト空間Hの非空閉凸集合C上で定義された写像$A : C \to H$について，

$$VI(C,A) = F(P_C(I - \mu A)) \quad \text{for all } \mu > 0$$

を証明せよ．ただし，P_CはHからCへの距離射影である．

7. Cを実ヒルベルト空間Hの非空部分集合とする．また，$A : C \to H$をη–strongly monotoneかつK–Lipschitz連続写像とする．ただし，$\eta \leq K$である．このとき，$\mu \in \left(0,\ \frac{2\eta}{K^2}\right)$に対して，写像$I - \mu A$は縮小写像になる．このことを示せ．

8. 次の定理を証明せよ.

定理. Cを実ヒルベルト空間Hの非空閉凸集合, $A : C \to H$をη-strongly monotoneかつK-Lipschitz連続写像とする. ただし, $\eta \leq K$である. また, $\mu \in \left(0, \dfrac{2\eta}{K^2}\right)$とする. このとき, 次の(1)と(2)が成り立つ.

(1) 変分不等式問題の解集合$VI(C,A)$は, ただ一つの要素x^*を持つ.

(2) x_1を任意のCの要素とし, ルール

$$x_{n+1} = P_C(I - \mu A)x_n$$

により点列$\{x_n\}$を逐次的に定めると, $x_n \to x^*$となる. ここで, x^*は(1)で存在を保証されている変分不等式の解であり, P_CはHからCへの距離射影である.

9. 問題8の定理において, もし$C = H$なら, 問題2より変分不等式問題の解集合$VI(C,A)$はAの零点集合$A^{-1}0$に一致する. この場合, 定理はどのように記述できるか? 正確に述べてみよ.

10. 問題8の定理の適用例や応用問題を自分で考え, それに解答を与えよ.