An alternative proof for the Banach contraction principle

$$\boxed{\begin{array}{l} \{a_n\} \subset [0,\infty) \\ \sum_{n=1}^{\infty} a_n < \infty \\ \Rightarrow \sum_{k=n}^{\infty} a_k \to 0 \ (n\to\infty) \end{array}}$$

Proof

Define $S_n = \sum_{k=1}^{n} a_k \in \mathbb{R}$.

As $\sum_{n=1}^{\infty} a_n$ is convergent,

$$S_n \to \sum_{n=1}^{\infty} a_n \in \mathbb{R}.$$

It holds that

$$\begin{aligned} &\sum_{n=1}^{\infty} a_n - \sum_{k=1}^{n-1} a_k \\ &= (a_1 + a_2 + a_3 + \cdots) \\ &\qquad - (a_1 + a_2 + \cdots + a_{n-1}) \\ &= (a_1 - a_1) + (a_2 - a_2) + \cdots + (a_{n-1} - a_{n-1}) \\ &\qquad + a_n + a_{n+1} + \cdots \\ &= \sum_{k=n}^{\infty} a_k \end{aligned}$$

Thus, we have

$$\sum_{k=n}^{\infty} a_k = \sum_{n=1}^{\infty} a_n - \sum_{k=1}^{n-1} a_k$$

$$= \sum_{n=1}^{\infty} a_n - S_{n-1} \to 0$$

as $n \to \infty$

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- The assumption that $\sum_{n=1}^{\infty} a_n < \infty$ is indispensable.

ex

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

In this case, $\sum_{k=n}^{\infty} \frac{1}{k} = \infty \quad \forall n \in \mathbb{N}$

$$\therefore \sum_{k=n}^{\infty} \frac{1}{k} \not\to 0 \ (n \to \infty).$$

$f: [1,\infty) \to [0,\infty)$

continuous,

monotone decreasing

$\Rightarrow$ Equivalent

① $\sum_{n=1}^{\infty} f(n) < \infty$

② $\int_1^\infty f(x)\,dx < \infty$

Proof.

As f is monotone decreasing,

$n \leqq x \leqq n+1$

$\Rightarrow f(n) \geqq f(x) \geqq f(n+1)$.

Therefore,

$$\begin{aligned} f(n) &= \int_n^{n+1} f(n)\,dx \\ &\geqq \int_n^{n+1} f(x)\,dx \\ &\geqq \int_n^{n+1} f(n+1)\,dx \\ &= f(n+1) \qquad \forall n \in \mathbb{N}. \end{aligned}$$

It follows that

$$\sum_{n=1}^{N-1} f(n) \geq \int_1^N f(x)\,dx$$

$$\geq \sum_{n=2}^{N} f(n) \quad {}^{\forall}N \in \mathbb{N}.$$

Thus, it holds that ① $\Leftrightarrow$ ②. //

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^s} \begin{cases} = \infty & \text{if } s \leq 1 \\ \in \mathbb{R} & \text{if } s > 1 \end{cases}}$$

Proof

(i) $s = 1$ OK

(ii) $s < 1$

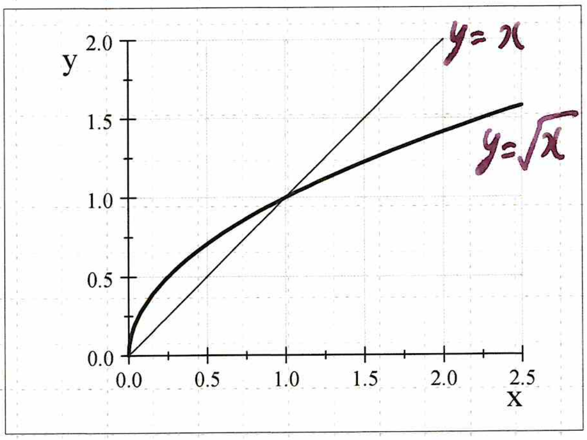
In this case, $n \geq n^s$, where $n \in \mathbb{N}$.

Therefore, $\frac{1}{n} \leq \frac{1}{n^s}$.

We obtain

$$\infty = \sum_{n=1}^{\infty} \frac{1}{n} \leq \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This indicates that $\sum_{n=1}^{\infty} \frac{1}{n^s} = \infty$. ⌟



$\sqrt{n} \leq n \quad \forall n \in \mathbb{N}.$

(iii) $s > 1$

We prove that $\int_1^{\infty} \frac{1}{x^s} \, dx \in \mathbb{R}$.

It follows that

$$\int_1^{\infty} \frac{1}{x^s} \, dx$$

$$= \lim_{k \to \infty} \int_1^{k} \frac{1}{x^s} \, dx$$

$$= \lim_{k \to \infty} \int_1^{k} x^{-s} \, dx$$

$$= \lim_{k \to \infty} \left[\frac{1}{1-s} x^{1-s} \right]_1^k$$

$$= \lim_{k \to \infty} \left(\frac{1}{1-s} \frac{1}{k^{\underset{>0}{\boxed{s-1}}}} - \frac{1}{1-s} \right)$$

$$= \frac{1}{s-1} \in \mathbb{R}$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^s} \in \mathbb{R}$ if $s > 1$. //

$$\boxed{\log x \overset{①}{<} x^{\frac{1}{2}} < x < x^2 \overset{②}{<} e^x}$$

"stronger"

①: $$\lim_{x\to\infty} \frac{x^{\frac{1}{2}}}{\log x}$$

l'Hôpital's rule

$$= \lim_{x\to\infty} \frac{\frac{1}{2}x^{-\frac{1}{2}}}{\frac{1}{x}}$$

$$= \frac{1}{2}\lim_{x\to\infty} x^{\frac{1}{2}} = \infty.$$

②: $$\lim_{x\to\infty} \frac{e^x}{x^2}$$

$$= \lim_{x\to\infty} \frac{e^x}{2x}$$

$$= \lim_{x\to\infty} \frac{e^x}{2}$$

$$= \infty.$$

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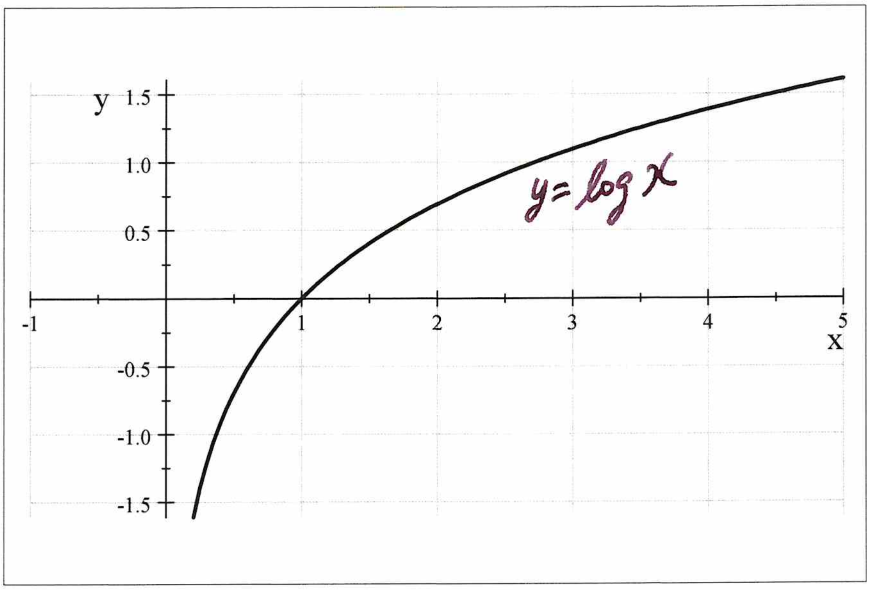
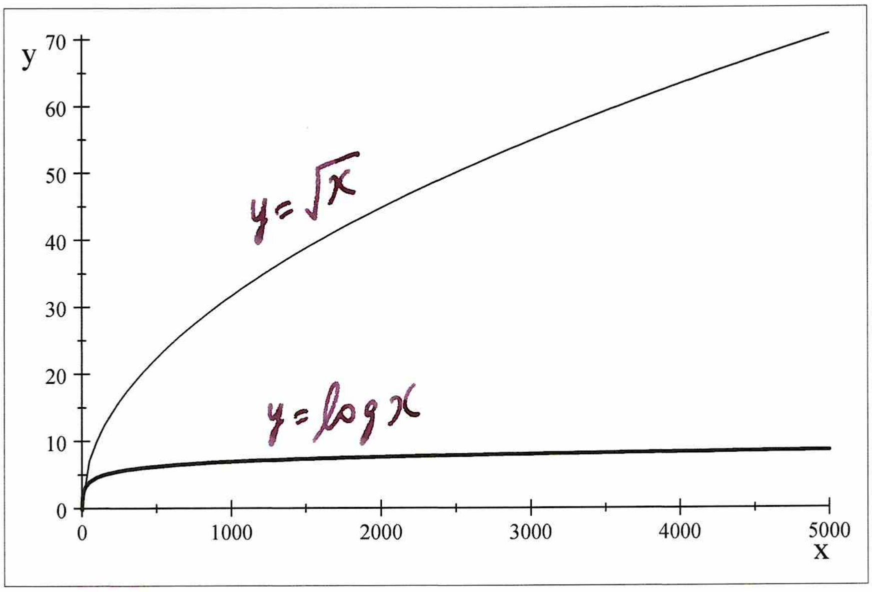
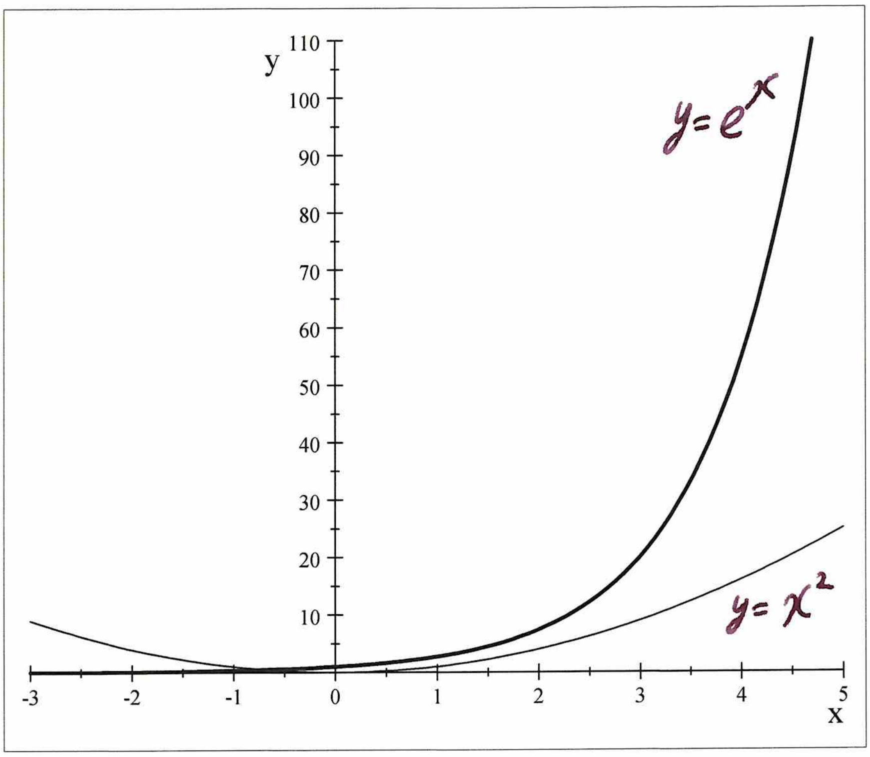
$y = \log x$



$y = \log x$
$y = \sqrt{x}$



$y = e^x$
$y = x^2$



$$\boxed{\lim_{x \to 0} \sqrt{x} \log x = (?)}$$

$$\lim_{x \to 0} -\sqrt{x} \log x$$

$$= \lim_{x \to 0} \frac{\log x}{-\frac{1}{\sqrt{x}}}$$

$$= \lim_{x \to 0} \frac{\log x}{-x^{-\frac{1}{2}}}$$

$$= \lim_{x \to 0} \frac{\frac{1}{x}}{\frac{1}{2} x^{-\frac{3}{2}}}$$

$$= \lim_{x \to 0} 2 \cdot x^{\frac{1}{2}} = 0.$$

$$\therefore \lim_{x \to 0} \sqrt{x} \log x = 0.$$

Th

X CMS
$T: X \to X$ r-contraction $(0 < r < 1)$
$\Rightarrow \exists^1 x^* \in F(T) : {}^\forall x \in X,\ T^n x \to x^*$

Proof.

⟨Existence⟩

Let $x \in X$ and define $x_n = T^n x$ $(n \in \mathbb{N} \cup \{0\})$.

We show that $\{x_n\}$ is a Cauchy sequence.

W.l.g., assume that $x_n \neq x_{n+1}$ $(n \in \mathbb{N} \cup \{0\})$.

If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then

$$x_{n+2} = Tx_{n+1} = Tx_n = x_{n+1}.$$

$$\therefore x_n = x_{n+1} = x_{n+2} = \cdots$$

As we assume that $x_n \neq x_{n+1}$,

$$d(x_n, x_{n+1}) > 0.$$

Therefore, $\log d(x_n, x_{n+1}) \in \mathbb{R}$ exists.

It holds that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$
$$\leq r\, d(x_{n-1}, x_n).$$

We have

$$\log d(x_n, x_{n+1}) \leq \log d(x_{n-1}, x_n) + \log r$$
$$\leq \log d(x_{n-2}, x_{n-1}) + 2\log r$$
$$\cdots$$
$$\leq \log d(x_0, x_1) + n\cdot \underbrace{\log r}_{<0} \quad -(*)$$
$$\longrightarrow -\infty \; (n \to \infty).$$

This means that $d(x_n, x_{n+1}) \to 0$. $-(**)$

It follows that

$$\sqrt{d(x_n, x_{n+1})}\left(\log d(x_n, x_{n+1}) - \log d(x_0, x_1)\right)$$
$$\leq n \log r \cdot \sqrt{d(x_n, x_{n+1})} < 0.$$

From $(**)$, $n\sqrt{d(x_n, x_{n+1})} \to 0 \; (n \to \infty)$.

$\therefore \exists n_0 \in \mathbb{N} : n \geq n_0 \Rightarrow n\sqrt{d(x_n, x_{n+1})} < 1.$

$\therefore n \geq n_0 \Rightarrow d(x_n, x_{n+1}) < \dfrac{1}{n^2}.$

Let $m \geqq n \geqq n_0$.

It is true that

$$d(x_n, x_m) \leqq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m)$$

$$\leqq \sum_{k=n}^{\infty} \frac{1}{k^2} < \infty.$$

Thus, we obtain

$$d(x_n, x_m) \to 0 \quad (m, n \to \infty).$$ ┘

As X is complete, $\exists x^* \in X : x_n \to x^*$. —(*3)

We demonstrate that $\underline{x^* \in F(T)}$.

We have that

$$Tx^* = T(\lim x_n) \quad \leftarrow (*3)$$

$$= \lim Tx_n \quad \text{) } T\text{: continuous}$$

$$= \lim x_{n+1} = x^*.$$ ┘

<Uniqueness>

Let $x^*, y^* \in F(T)$.

It follows that

$$d(x^*, y^*)$$
$$= d(Tx^*, Ty^*)$$
$$\leqq r\, d(x^*, y^*).$$

$\therefore (1-r)\, d(x^*, y^*) \leqq 0.$

As $0 < r < 1$, we obtain

$$d(x^*, y^*) \leqq 0.$$

$\therefore x^* = y^*.$

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Define $F:(0,\infty)\to\mathbb{R}$ by
$$F(x)=\log x \quad \forall x\in(0,\infty).$$

<u>Properties of F we used.</u>

(F1) F is monotone increasing.

(F2) $F(x)\to-\infty \Rightarrow x\to 0$

(F3) For $k\in(0,1)$,
$$x^k F(x)\to 0 \text{ as } x\to 0$$

(F4) $F(nd)=F(n)+F(d)$

An alternative proof for the Banach contraction principle

1. $\sum_{n=1}^{\infty}\frac{1}{n}=\infty$の証明を復習せよ.（補論第1講を参照すればよい.）

2. 非負の実数からなる数列$\{a_n\}$に対して，$\sum_{n=1}^{\infty}a_n<\infty$とする. このとき，$\sum_{k=n}^{\infty}a_k\to 0$ (as $n\to\infty$) となることを示せ.

3. $f:[1,\infty)\to[0,\infty)$を連続な単調増加関数とする. このとき，次の2条件は同値であることを証明せよ.
(1) $\sum_{n=1}^{\infty}f(n)<\infty$,
(2) $\int_1^{\infty}f(x)dx<\infty$.

4. 次を証明せよ.

$$\sum_{n=1}^{\infty}\frac{1}{n^s}\begin{cases}=\infty & \text{if } s\le 1,\\ <\infty & \text{if } s>1.\end{cases}$$

5. ロピタルの定理を用いて，次を証明せよ.

$$(1)\quad \lim_{x\to\infty}\frac{\sqrt{x}}{\log x}=\infty$$
$$(2)\quad \lim_{x\to\infty}\frac{e^x}{x^2}=\infty$$

6. ロピタルの定理を用いて，

$$\lim_{x\to 0}\sqrt{x}\log x=0$$

を示せ.

7. 縮小写像の定義を確認し，それが連続であることを証明せよ.

8. 縮小写像の不動点定理を対数関数$F(x)=\log x$を用いて証明せよ. その際，次の対数関数の性質をどこで用いたか，確認せよ.

(F1) F is monotone increasing,
(F2) $F(x)\to-\infty\Rightarrow x\to 0$,
(F3) For $k\in(0,1)$, $x^kF(x)\to 0$ as $x\to 0$,
(F4) $F(rd)=F(r)+F(d)$.

9. 次の論文を読み，その内容をゼミで報告せよ.
Wardowski, D., “Fixed points of a new type of contractive mappings in complete metric spaces,” Fixed Point Theory Appl. **2012**, 94 (2012).