Mappings

Def

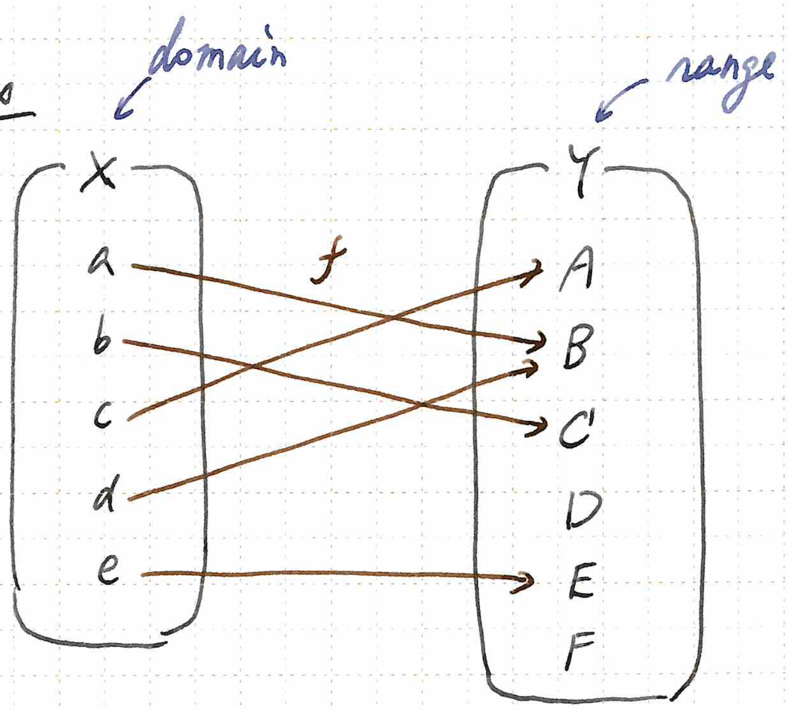
X, Y sets, $\neq \emptyset$

$f : X \to Y$ <u>mapping</u>

(<u>function</u>, <u>operator</u>)

$\Longleftrightarrow \forall x \in X, \exists^1 y \in Y : y = f(x)$

ex

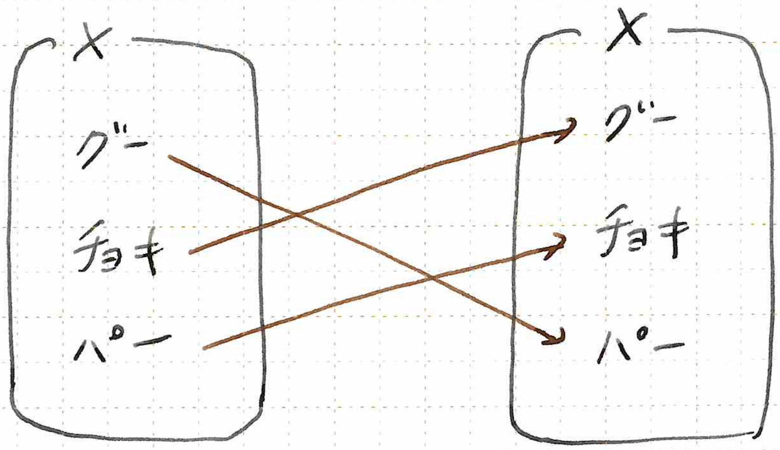


Def

$f, g : X \to Y$

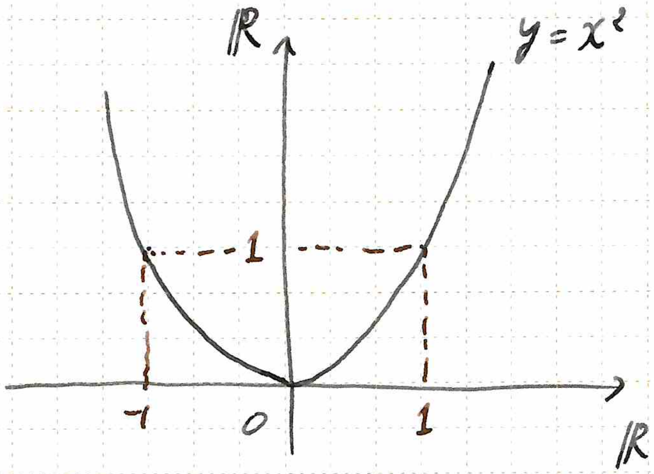
$f = g \Longleftrightarrow \forall x \in X, f(x) = g(x)$

Special case : $X = Y$



Special case : $X = Y = \mathbb{R}$

$y = x^2$



Def

$f : X \to Y$ 1-1 mapping

$\iff x \neq y \Rightarrow f(x) \neq f(y)$

$\iff f(x) = f(y) \Rightarrow x = y$

对偶 contraposition

$A \Rightarrow B$

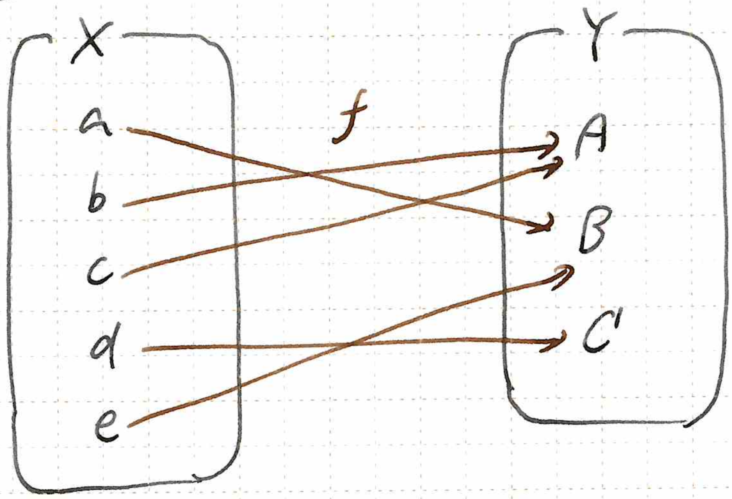
$\iff \neg B \Rightarrow \neg A$

$\neg$: 否定

$f : X \to Y$ is *not* a 1-1 mapping.

$$\iff \exists\, x, y \in X : \begin{cases} x \neq y \\ f(x) = f(y) \end{cases}$$

ex



ex

$X = Y = \mathbb{R}$

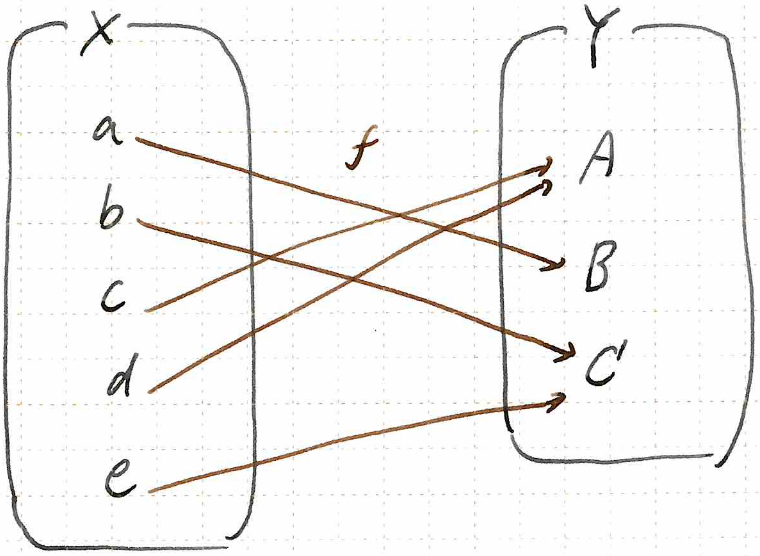
$f(x) = x^2$

Def.

$f: X \to Y$ <u>onto</u>

$\Longleftrightarrow \forall y \in Y, \exists x \in X : y = f(x)$

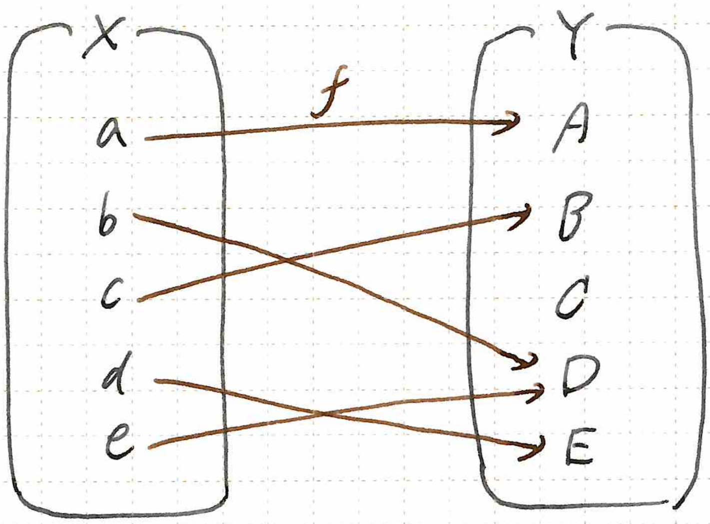
ex



$f: X \to Y$ is not an onto mapping.

$\iff \exists y \in Y : {}^\forall x \in X,\ y \neq f(x)$

ex

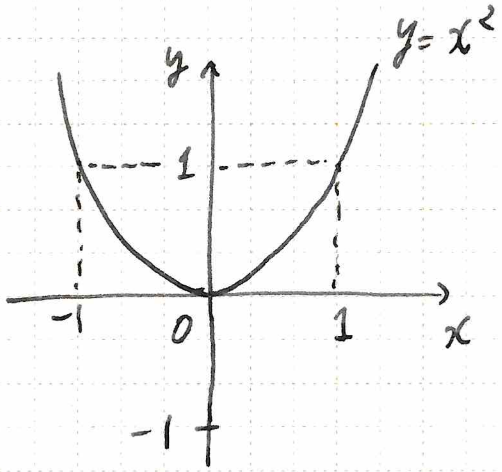


$\exists y (= C) \in Y : {}^\forall x \in X,\ y \neq f(x)$

ex

$X = Y = \mathbb{R}$

$f(x) = x^2$



f is not onto.

i.e. $\exists\, y (= -1) \in Y : {}^{\forall} x \in X,\ y \neq f(x)$

ex

$X = \mathbb{R}$

$Y = [0, \infty)$

$f(x) = x^2$

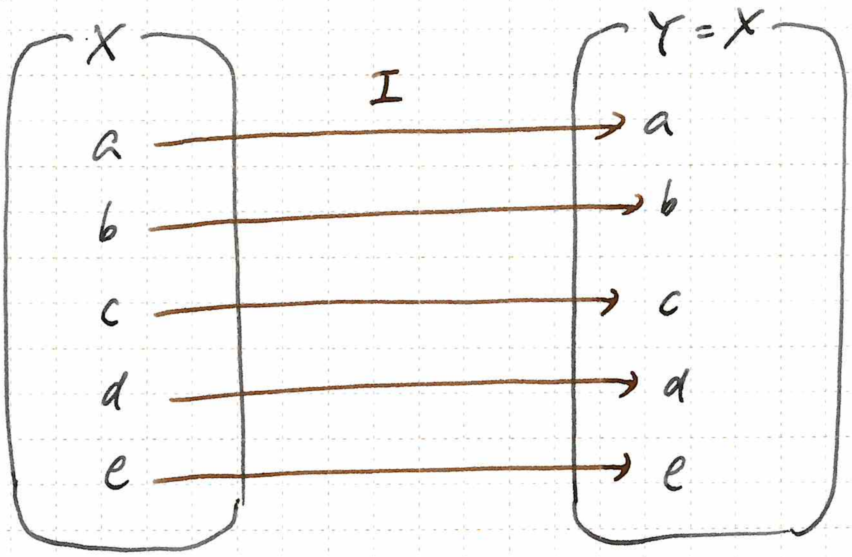
Then, f is an onto mapping.

Def

$I : X \to X$ *identity mapping*

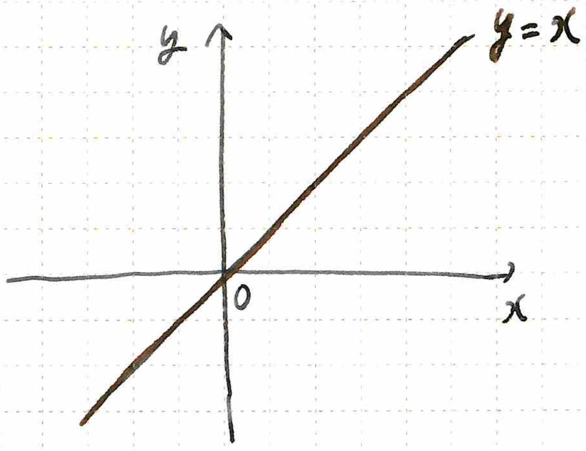
$\iff \forall x \in X,\ I(x) = x$

ex



ex

$I : \mathbb{R} \to \mathbb{R}$

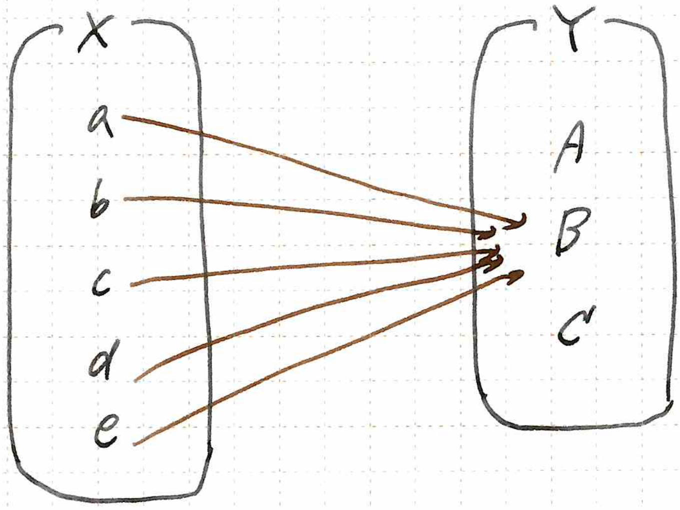


Def

$f: X \to Y$ constant mapping

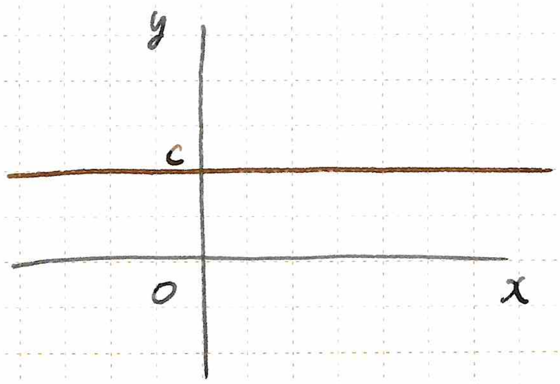
$\iff {}^{\exists} y \in Y : {}^{\forall} x \in X,\ y = f(x)$

ex



ex

$f: \mathbb{R} \to \mathbb{R}$



$f: X \to Y$ constant

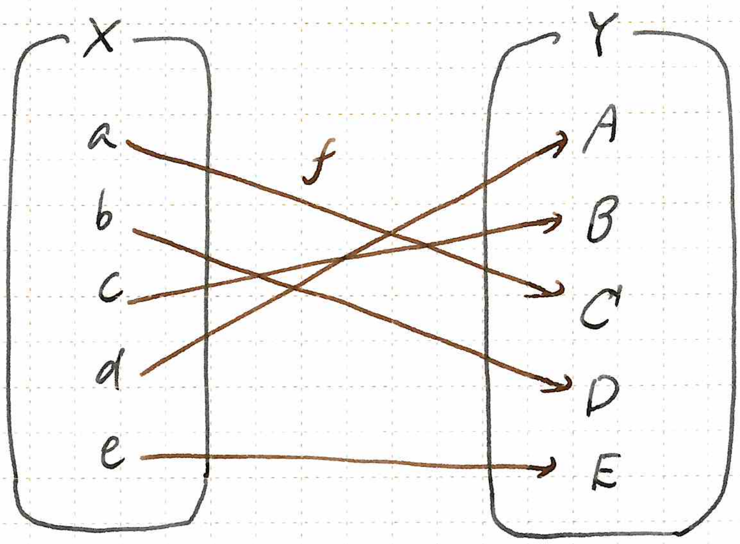
$\Longleftrightarrow \underline{\exists y \in Y : \forall x \in X,}\ y = f(x)$

比較

$f: X \to Y$ onto

$\Longleftrightarrow \underline{\forall y \in Y, \exists x \in X :}\ y = f(x)$

$f: X \to Y$ 1-1, onto



Then, $\forall y \in Y, \exists^1 x \in X : y = f(x)$

$x = f^{-1}(y)$

$f^{-1} : Y \to X$ inverse mapping of f. 逆関数

$f^{-1}(A) = d$

$f^{-1}(B) = c$

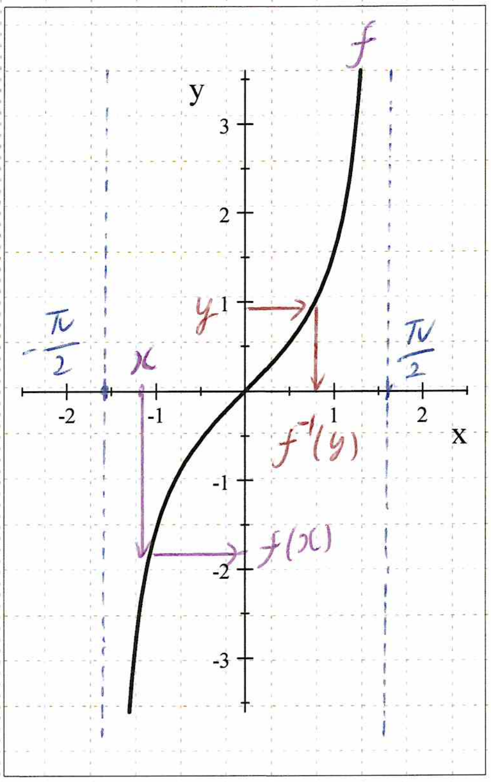
$f^{-1}(C) = a$

$f^{-1}(D) = b$

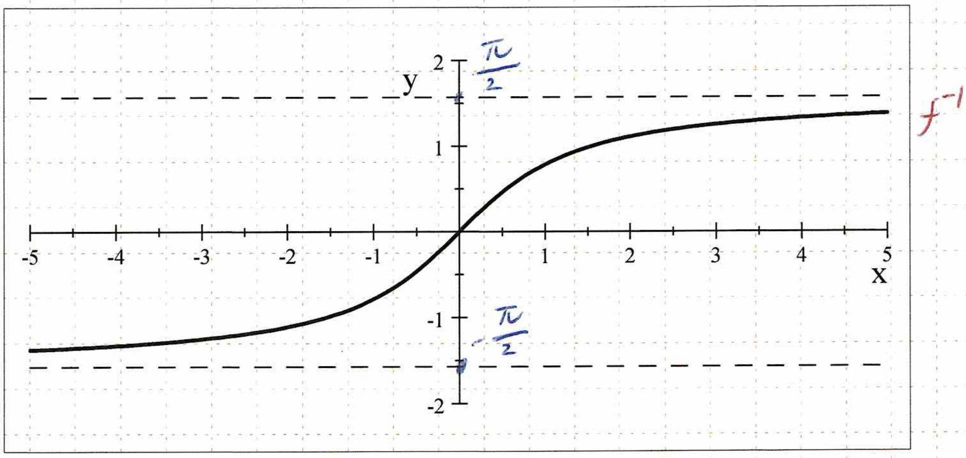
$f^{-1}(E) = e$

$y = f(x) = \tan x$

$f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$



$y = \arctan x \ (= Tan^{-1} x)$



X, Y, Z sets, $\neq \phi$

$f : X \to Y$

$g : Y \to Z$

Def.

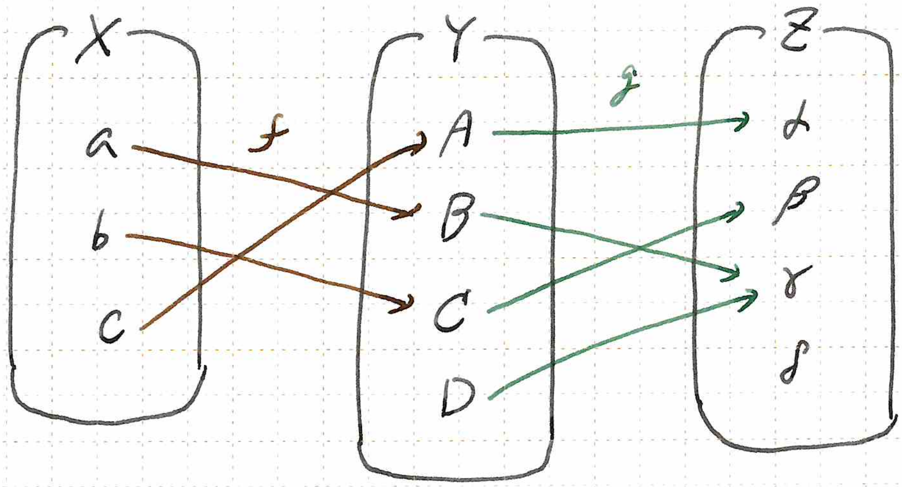
$g \circ f : X \to Z$ defined by

$$(g \circ f)(x) = g(f(x)) \quad {}^{\forall} x \in X$$

composite function of f and g

合成関数

ex



$$(g \circ f)(a)$$
$$= g(f(a))$$
$$= g(B)$$
$$= \gamma \in Z$$

$$(g \circ f)(b)$$
$$= g(f(b))$$
$$= g(C)$$
$$= \beta \in Z$$

$$(g \circ f)(c) = ?$$

$X = Y = Z$

$f, g : X \to X$

Then, $g \circ f : X \to X$

$f \circ g : X \to X$

$\boxed{g \circ f \neq f \circ g}$ in general.

<u>ex</u>

$X = \mathbb{R}$

$f(x) = -x$

$g(x) = x^2$

Then, $(g \circ f)(x) = g(f(x))$

$= g(-x)$

$= (-x)^2 = x^2,$

$(f \circ g)(x) = f(g(x))$

$= f(x^2)$

$= -x^2.$

$\therefore\ g \circ f \neq f \circ g$

$X, Y \neq \phi$

$f : X \to Y$

$A \subset X$

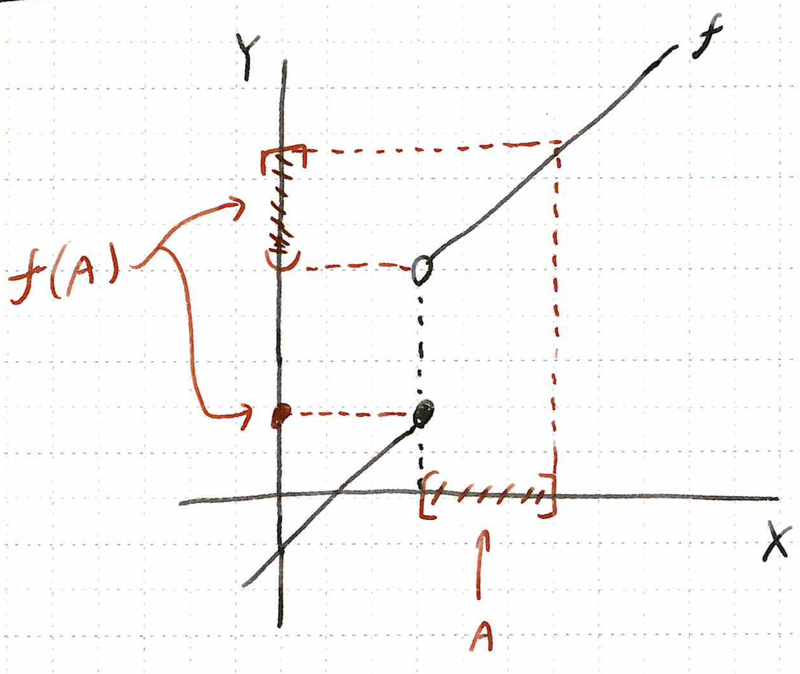
Def.

$$f(A) = \{ y \in Y \mid \exists x \in A : y = f(x) \}$$

$$= \{ f(x) \in Y \mid x \in A \}$$

image of A by f

ex

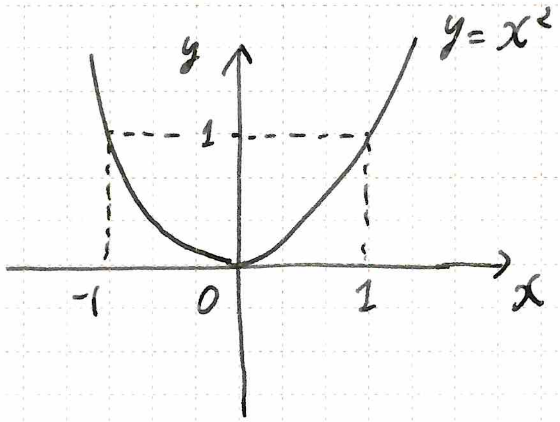


ex

$X = Y = \mathbb{R}$

$f : X \to Y$ defined by

$f(x) = x^2 \quad \forall x \in X$



Let
$$\begin{cases} A_1 = [0,1] & \subset X \\ A_2 = [-1,1] & \subset X \\ A_3 = \mathbb{R}. & \subset X \end{cases}$$

Then,
$$\begin{cases} f(A_1) = [0,1] & \subset Y \\ f(A_2) = [0,1] & \subset Y \\ f(A_3) = [0,\infty) & \subset Y. \end{cases}$$

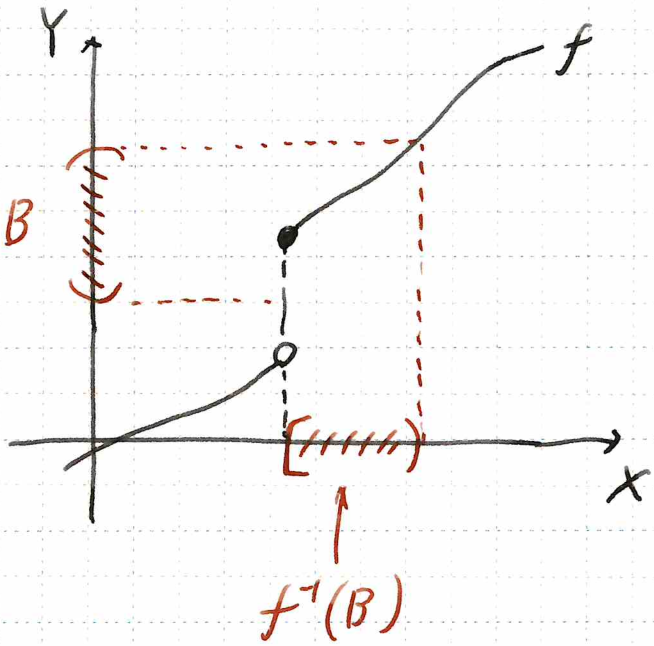
$B \subset Y$

Def

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}$$

inverse image of B by f

ex



- $y \in f(A)$, where $A \subset X$
  $\Leftrightarrow \exists x \in A : y = f(x)$
- $x \in f^{-1}(A)$, where $A \subset Y$
  $\Leftrightarrow f(x) \in A$

ex

$X = Y = \mathbb{R}$

$f(x) = x^2$

Let

$B_1 = [0,1] \subset Y$

$B_2 = [0, \infty)$

$B_3 = (-\infty, 0]$

$B_4 = \mathbb{R}$

$B_5 = \{1\}$

Then,

$f^{-1}(B_1) = [-1, 1] \subset X$

$f^{-1}(B_2) = \mathbb{R}$

$f^{-1}(B_3) = \{0\}$

$f^{-1}(B_4) = \mathbb{R}$

$f^{-1}(B_5) = \{-1, 1\}$.

$X = \{a, b\}$
$Y = \{1, 2, 3\}$

$(a, 1), (b, 3), \cdots$ <u>ordered pair</u>

順序対

$(a, 3) \neq (3, a)$

cf. $\{a, 3\} = \{3, a\}$

<u>Def</u>

$X, Y \neq \emptyset$

$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$

<u>Cartesian product</u>

カルテシアン・プロダクト

直積

ex

$X = \{a, b\}$

$Y = \{1, 2, 3\}$

Then, $X \times Y = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$

$X^2 = X \times X$

ex

$X = \{a, b\}$

Then, $X^2 = \{(a,a), (a,b), (b,a), (b,b)\}$

ex

$\mathbb{R}^2 = \{(x,y) \mid x, y \in \mathbb{R}\}$

$(2,-1), (\pi,-e), (\frac{1}{2},-4) \in \mathbb{R}^2$

Def

$X, Y \neq \emptyset$

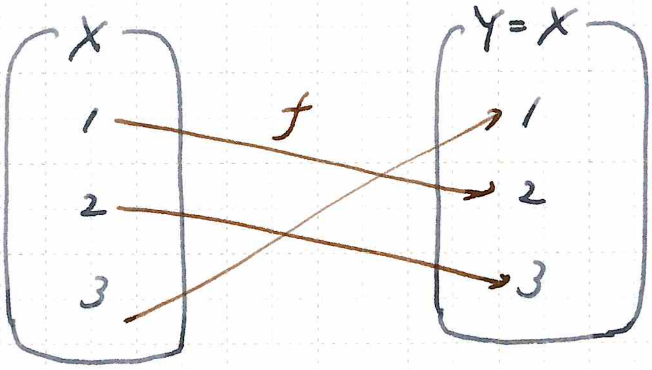
$f : X \to Y$

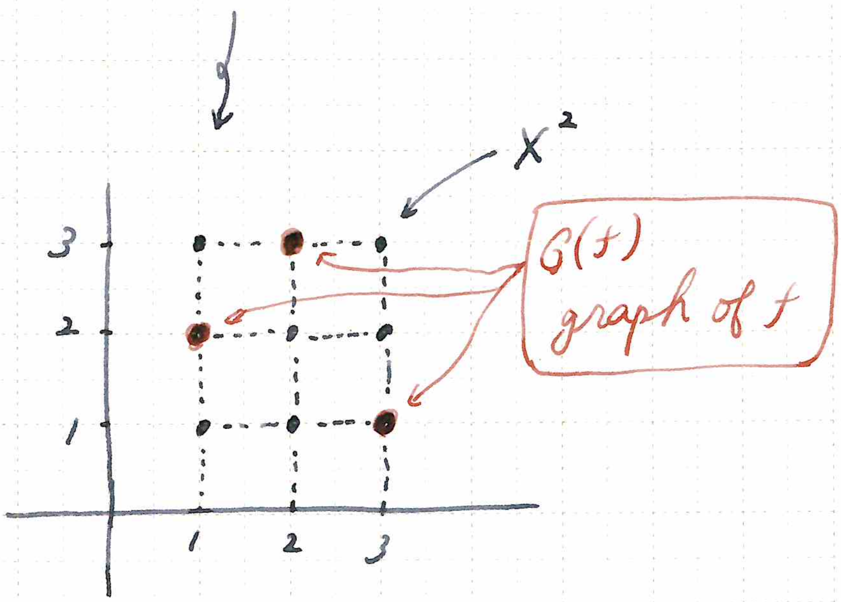
$G(f) = \{ (x, y) \in X \times Y \mid y = f(x) \}$

<u>graph of f</u>

ex

$X = Y = \{1, 2, 3\}$



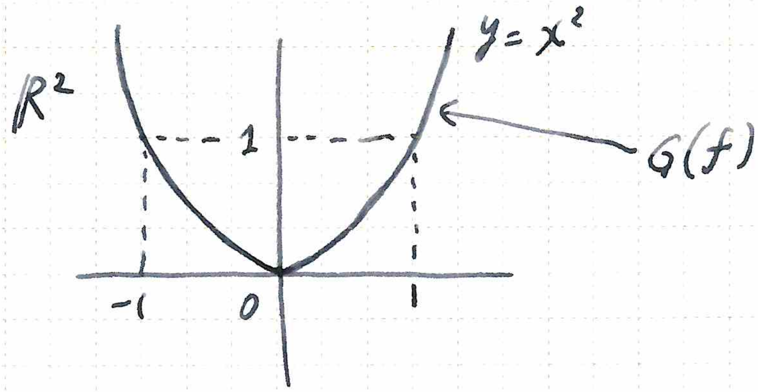


ex

$X = Y = \mathbb{R}$

$f : X \to Y$ defined by

$f(x) = x^2 \quad \forall x \in \mathbb{R}$



※ $f : X \to Y$

- $\operatorname{Im} f = f(X) \subset Y$ image
- $f^{-1}(Y) \subset X$ inverse image
- $G(f) \subset X \times Y$ graph

Mappings

1. 1対1の写像と上への写像の定義を述べよ．また，1対1ではない写像と上への写像ではない写像は，論理的にどう表現されるか？

2. 次のタイプの写像を考えてみよ．
(1) 1対1かつ上への写像
(2) 1対1写像でも上への写像でもない写像
(3) 1対1だが，上への写像ではない写像
(4) 1対1ではないが，上への写像

3. 3つの集合X, Y, Zを

$$X = \{a, b, c\},$$
$$Y = \{A, B, C\},$$
$$Z = \{\alpha, \beta, \gamma\}$$

とする．また，写像$f : X \to Y$を

$$f(a) = C,$$
$$f(b) = A,$$
$$f(c) = B,$$

写像$g : Y \to Z$を

$$g(A) = \alpha,$$
$$g(B) = \gamma,$$
$$g(C) = \beta$$

とする．
(1) 合成写像$g \circ f$を求めよ．
(2) $g \circ f$の逆写像$(g \circ f)^{-1}$を求めよ．この写像の定義域と値域はどこか？
(3) fとgの逆写像f^{-1}, g^{-1}を求めよ．
(4) g^{-1}とf^{-1}の合成写像$f^{-1} \circ g^{-1}$を求め，それが(2)で求めた$(g \circ f)^{-1}$に等しいことを確認せよ．

4. X, Y, Zを空でない集合とする．2つの写像$f : X \to Y$と$g : Y \to Z$について，以下を証明せよ．
(1) fとgが1対1ならば，$g \circ f$も1対1である．
(2) fとgが上への写像ならば，$g \circ f$も上への写像である．

5. 関数$f : \mathbb{R} \to \mathbb{R}$を

$$f(x) = \begin{cases} -x + 3 & \text{if } x < 1 \\ -x + 2 & \text{if } x \geq 1 \end{cases}$$

とする．(1) $f([0, 1])$ (2) $f([-1, 2])$ (3) $f^{-1}([0, 2])$ (4) $f^{-1}((0, \frac{3}{2}))$ を求めよ．

6. 関数$f(x) = x^2$について，(1) $f(f^{1}([-1, 1]))$ (2) $f^{-1}(f([0, 1]))$ を求めよ．

7. 関数$f(x) = \cos x$について，$f^{-1}(f(\{0\}))$ を求めよ.

8. $X = \{1, 2, 3, 4\}$とする. 以下の問いに答えよ.
(1) 直積集合X^2を求めよ.
(2) 写像$f : X \to X$を

$$f(1) = 2, \quad f(2) = 4, \quad f(3) = 1, \quad f(4) = 2$$

と定義する. この写像のグラフを描け.

9. 関数$f(x) = 1/x$と$g(x) = x$のグラフは下図のようになる. このとき, $f + g$のグラフの概形を“関数の和”の定義に従って手で描いてみよ.