Images and inverse images

Review

$X, Y \neq \emptyset$

$f : X \to Y$

$A \subset X$

$B \subset Y$

Def

$$f(A) = \{ y \in Y \mid \exists x \in A : y = f(x) \} \subset Y$$

$$= \{ f(x) \in Y \mid x \in A \}$$

<u>image</u> of A by f

$$f^{-1}(B) = \{ x \in X \mid f(x) \in B \} \subset X$$

<u>inverse image</u> of B by f

- $y \in f(A)$

  $\Leftrightarrow \exists x \in A : y = f(x)$

- $x \in f^{-1}(B)$

  $\Leftrightarrow f(x) \in B$

$A_\mu \ (\mu \in M)$

- $x \in \bigcup_{\mu \in M} A_\mu$

  $\Leftrightarrow \exists \mu \in M : x \in A_\mu$

- $x \in \bigcap_{\mu \in M} A_\mu$

  $\Leftrightarrow \forall \mu \in M, x \in A_\mu$

$$\boxed{\begin{array}{l} X, Y \neq \emptyset \\ f: X \to Y \\ A \subset B \subset X \\ \Rightarrow f(A) \subset f(B) \end{array}}$$

<u>Proof</u>

Let $y \in f(A)$.

Then, $\exists x \in A : y = f(x)$.

As $A \subset B$, we have

$\exists x \in B : y = f(x)$.

This implies that $y \in f(B)$.

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ex

$X = Y = \mathbb{R}$

$f: X \to Y$ defined by

$f(x) = x^2 \quad \forall x \in X = \mathbb{R}$

$A = [0, 1] \subset X$

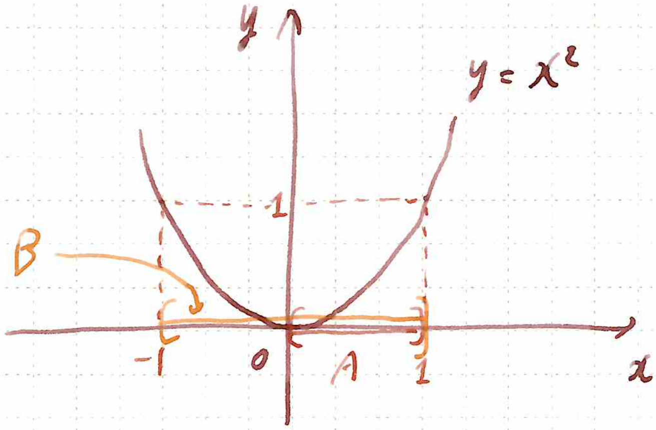
$B = [-1, 1] \subset X$

Then, $A \subset B$

$f(A) = [0,1] \subset Y$

$f(B) = [0,1] \subset Y$

$\therefore f(A) \subset f(B)$



$X, Y \neq \emptyset$

$f : X \to Y$

$A \subset B \subset Y$

$\Rightarrow f^{-1}(A) \subset f^{-1}(B)$

<u>Proof</u>

Let $x \in f^{-1}(A)$.

i.e. $f(x) \in A$

As $A \subset B$, we have $f(x) \in (A \subset) B$.

Therefore, $x \in f^{-1}(B)$.

$\therefore f^{-1}(A) \subset f^{-1}(B)$.

//

ex

$X = Y = \mathbb{R}$

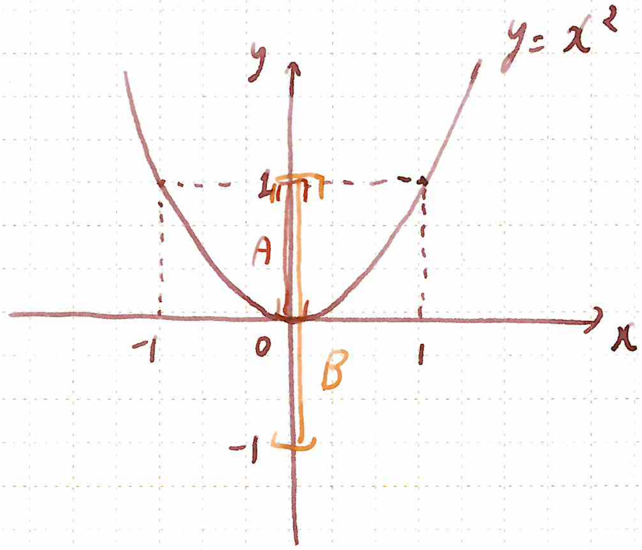
$f(x) = x^2$

$A = [0, 1] \subset Y$

$B = [-1, 1] \subset Y$

Then, $\begin{cases} A \subset B \\ f^{-1}(A) = f^{-1}(B) = [-1, 1] \subset X. \end{cases}$

Thus, $f^{-1}(A) \subset f^{-1}(B)$.



$X, Y \neq \emptyset$

$f : X \to Y$

$A_\mu \subset X \quad (\mu \in M)$

$\Longrightarrow f\left(\bigcup_\mu A_\mu\right) = \bigcup_\mu f(A_\mu)$

<u>Proof.</u>

We have the desired result as follows:

$$
\begin{aligned}
& y \in \bigcup_\mu f(A_\mu) \\
\Longleftrightarrow\ & \exists \mu \in M : y \in f(A_\mu) \\
\Longleftrightarrow\ & \exists \mu \in M,\ x \in A_\mu : y = f(x). \\
\Longleftrightarrow\ & \exists x \in \bigcup_\mu A_\mu : y = f(x) \\
\Longleftrightarrow\ & y \in f\left(\bigcup_\mu A_\mu\right).
\end{aligned}
$$

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$X, Y \neq \emptyset$

$f: X \to Y$

$B_\mu \subset Y \quad (\mu \in M)$

$\Longrightarrow f^{-1}\left(\bigcup_\mu B_\mu\right) = \bigcup_\mu f^{-1}(B_\mu)$

Proof

$x \in \bigcup_\mu f^{-1}(B_\mu)$

$\Longleftrightarrow \exists \mu \in M: x \in f^{-1}(B_\mu)$

$\Longleftrightarrow \exists \mu \in M: f(x) \in B_\mu$

$\Longleftrightarrow f(x) \in \bigcup_\mu B_\mu$

$\Longleftrightarrow x \in f^{-1}\left(\bigcup_\mu B_\mu\right).$

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$$\boxed{\begin{array}{l} X, Y \neq \emptyset \\ f : X \to Y \\ A_\mu \subset X \ (\mu \in M) \\ \Rightarrow f\left(\bigcap_\mu A_\mu\right) \subset \bigcap_\mu f(A_\mu) \end{array}}$$

Proof.

Let $y \in f\left(\bigcap_\mu A_\mu\right)$.

i.e. $\exists x \in \bigcap_\mu A_\mu : y = f(x)$. — (*)

We show that $\underline{y \in \bigcap_\mu f(A_\mu)}$.

i.e. $\forall \mu \in M, \ y \in f(A_\mu)$.

i.e. $\forall \mu \in M, \ \exists x_\mu \in A_\mu : y = f(x)$.

From (*), $\exists x \in X : \forall \mu \in M, \begin{cases} x \in A_\mu \\ y = f(x). \end{cases}$

Thus, we obtain

$\forall \mu \in M, \ \exists x \in X : \begin{cases} x \in A_\mu \\ y = f(x). \end{cases}$

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ex

$X = Y = \mathbb{R}$

$f(x) = x^2$

$A_1 = [-1, 0] \subset X$

$A_2 = [0, 1] \subset X$

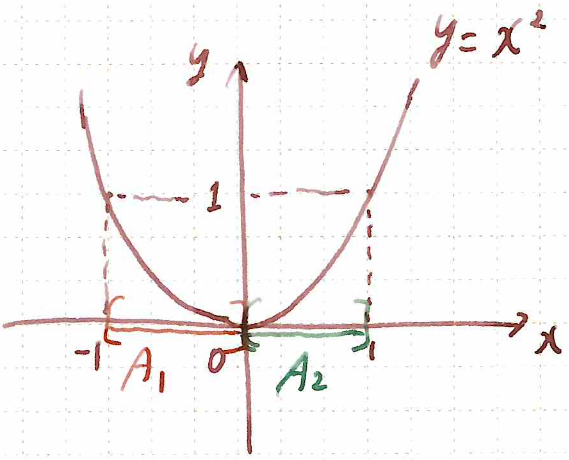
Then, $A_1 \cap A_2 = \{0\} \subset X$

$f(A_1 \cap A_2) = \{0\} \subset Y$

$f(A_1) = f(A_2) = [0, 1] \subset Y.$

Therefore,

$$\underset{\{0\}}{\underset{\shortparallel}{f(A_1 \cap A_2)}} \subset f(A_1) \cap f(A_2) = [0, 1].$$



$X, Y \neq \emptyset$

$f: X \to Y$

$B_\mu \subset Y \quad (\mu \in M)$

$\Longrightarrow f^{-1}\left(\bigcap_\mu B_\mu\right) = \bigcap_\mu f^{-1}(B_\mu)$

Proof.

$x \in f^{-1}\left(\bigcap_\mu B_\mu\right)$

$\Longleftrightarrow f(x) \in \bigcap_\mu B_\mu$

$\Longleftrightarrow \forall_{\mu \in M},\ f(x) \in B_\mu$

$\Longleftrightarrow \forall_{\mu \in M},\ x \in f^{-1}(B_\mu)$

$\Longleftrightarrow x \in \bigcap_\mu f^{-1}(B_\mu)$.

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ex

$X = Y = \mathbb{R}$

$f(x) = x^2$

$B_1 = [0, 1] \subset Y$

$B_2 = [-1, 0] \subset Y$

Then, $\begin{cases} B_1 \cap B_2 = \{0\} \subset Y \\ f^{-1}(B_1) = [-1, 1] \subset X \\ f^{-1}(B_2) = \{0\} \subset X. \end{cases}$

Therefore, $\begin{cases} f^{-1}(B_1 \cap B_2) = \{0\} \subset X \\ f^{-1}(B_1) \cap f^{-1}(B_2) = \{0\} \subset X. \end{cases}$

$\therefore f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2).$

$$\boxed{\begin{array}{l} X, Y \neq \emptyset \\ f : X \to Y \\ A \subset X \\ \Rightarrow A \subset f^{-1}(f(A)) \end{array}}$$

Proof.

Let $x \in A$.

We prove that $\underline{x \in f^{-1}(f(A))}$.

i.e. $f(x) \in f(A)$.

OK.

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ex

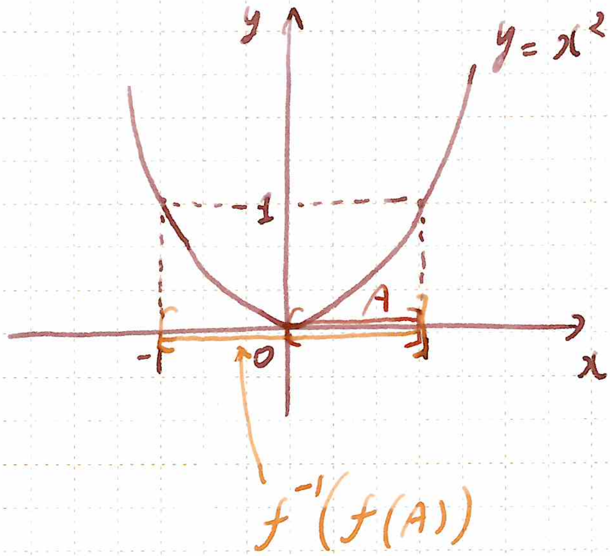
$X = Y = \mathbb{R}$

$f(x) = x^2$

$A = [0,1] \subset X$

Then, $\left(\begin{array}{l} f(A) = [0,1] \subset Y \\ f^{-1}(f(A)) = [-1,1] \subset X. \end{array} \right.$

Hence, $A \subset f^{-1}(f(A))$.



$$\boxed{\begin{array}{l} X, Y \neq \emptyset \\ f : X \to Y \\ B \subset Y \\ \Rightarrow f(f^{-1}(B)) \subset B \end{array}}$$

<u>Proof.</u>

Let $y \in f(f^{-1}(B))$.

Then, $\exists x \in f^{-1}(B) : y = f(x)$.

As $x \in f^{-1}(B)$, it follows that

$$f(x) \in B.$$

Therefore, $y = f(x) \in B$.

$\therefore f(f^{-1}(B)) \subset B$.

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ex

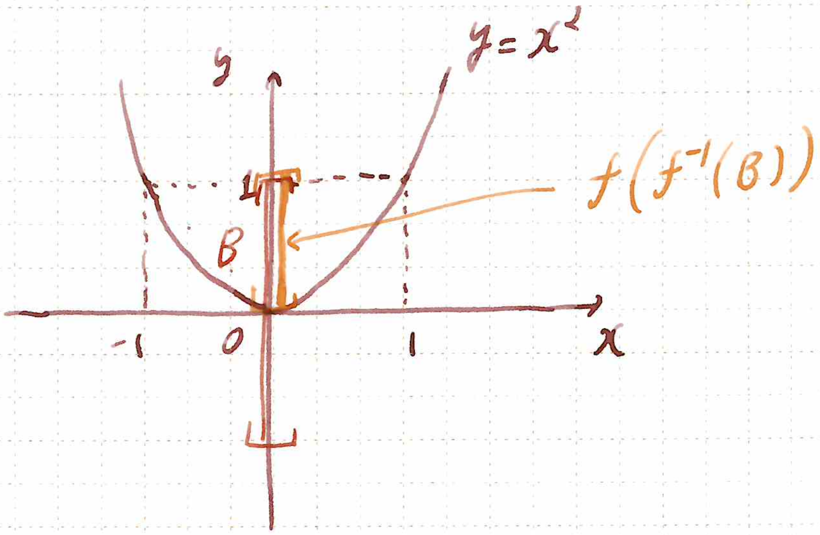
$X = Y = \mathbb{R}$

$f(x) = x^2$

$B = [-1, 1] \subset Y$

Then, $\begin{cases} f^{-1}(B) = [-1, 1] \subset X \\ f(f^{-1}(B)) = [0, 1] \subset Y \end{cases}$

$\therefore f(f^{-1}(B)) \subset B$



$X, Y \neq \phi$

$f: X \to Y$

$B \subset Y$

$\Rightarrow f^{-1}(B^c) = \left(f^{-1}(B)\right)^c$

Proof

$x \in \left(f^{-1}(B)\right)^c$

$\Leftrightarrow x \notin f^{-1}(B)$

$\Leftrightarrow f(x) \notin B$

$\Leftrightarrow f(x) \in B^c$

$\Leftrightarrow x \in f^{-1}(B^c)$.

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$X, Y \neq \emptyset$

$f : X \to Y$

$A \subset X$

$\Longrightarrow f(A^c) \neq (f(A))^c$

ex

$X = Y = \mathbb{R}$

$f(x) = x^2$

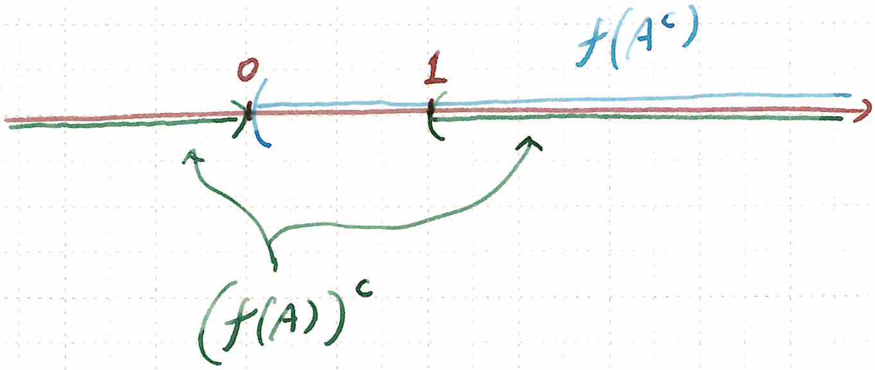
$A = [0, 1]$

Then,

$f(A^c) = (0, \infty)$

$f(A) = [0, 1]$

$(f(A))^c = (-\infty, 0) \cup (1, \infty)$



注意点

	合併	共通部分
像		注意！
逆像		

・二重に像（像と逆像）をとる操作

・像と補集合をとる操作の組み合わせ

Images and inverse images

XとYを空でない集合, $f : X \to Y$とする. 以下を証明するとともに, 例を用いて説明せよ(レジュメの例で構わない).

1. $A \subset B \subset X$のとき, $f(A) \subset f(B)$となる.

2. $A \subset B \subset Y$のとき, $f^{-1}(A) \subset f^{-1}(B)$となる.

3. $A_1, A_2 \subset X$のとき, $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$となる.

4. $B_\mu \subset Y\ (\mu \in M)$のとき, $f^{-1}\left(\bigcup_{\mu \in M} B_\mu\right) = \bigcup_{\mu \in M} f^{-1}(B_\mu)$となる.

5. $A_\mu \subset X\ (\mu \in M)$のとき, $f\left(\bigcap_{\mu \in M} A_\mu\right) \subset \bigcap_{\mu \in M} f(A_\mu)$となる.

6. $B_\mu \subset Y\ (\mu \in M)$のとき, $f^{-1}\left(\bigcap_{\mu \in M} B_\mu\right) = \bigcap_{\mu \in M} f^{-1}(B_\mu)$となる.

7. $A \subset X$のとき, $A \subset f^{-1}(f(A))$となる.

8. $B \subset Y$のとき, $f(f^{-1}(B)) \subset B$となる.

9. $B \subset Y$のとき, $f^{-1}(B^C) = (f^{-1}(B))^C$となる.