Fixed point theorems in complete metric spaces

X MS
$T: X \to X$ continuous
$\Rightarrow F(T)$ is closed in X.

Proof.
Let $\{x_n\} \subset F(T) : x_n \to x \in X$. — (*)
We show that $x \in F(T)$.
i.e. $x = Tx$
i.e. $d(x, Tx) \leqq 0$.

As T is continuous and $x_n \to x$,
it holds that $Tx_n \to Tx$. — (**)
Therefore,

$$d(x, Tx) \leqq \underbrace{d(x, x_n)}_{\substack{\to 0 \\ (*)}} + \underbrace{d(x_n, Tx_n)}_{\substack{= 0 \\ (\because)\ x_n \in F(T)}} + \underbrace{d(Tx_n, Tx)}_{\substack{\to 0 \\ (**)}} \to 0.$$

We obtain $d(x, Tx) \leqq 0$.
$\therefore x = Tx$. //

※ $F(T) = \{x \in X \mid Tx = x\}$.

V vector space
$T: V \to V$ linear
$\Rightarrow F(T)$ is a subspace in V.

<u>Proof.</u>

Let $x, y \in F(T)$; $\alpha, \beta \in K$ (scalar).

We show that $\alpha x + \beta y \in F(T)$.

i.e. $\alpha x + \beta y = T(\alpha x + \beta y)$.

It follows that

$$
\begin{aligned}
& T(\alpha x + \beta y) \\
& = \alpha Tx + \beta Ty && \text{) } T\text{: linear} \\
& = \alpha x + \beta y. && \text{) } x, y \in F(T).
\end{aligned}
$$

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Review

Def.

X, Y metric spaces (MSs)

$T: X \to Y$ <u>K-Lipschitz continuous</u>

$\Leftrightarrow \exists K > 0 : \forall x, y \in X,$

$$d(Tx, Ty) \leq K d(x, y)$$

$T: X \to Y$ Lipschitz continuous

$\Rightarrow T$: continuous

Def.

$T: X \to Y$ <u>a-contraction</u>

$\Leftrightarrow T$: a-Lipschitz with $a \in (0,1)$

$\Leftrightarrow \exists a \in (0,1) : \forall x, y \in X,$

$$d(Tx, Ty) \leq a d(x, y)$$

$T : X \to X$

$F(T) = \{ u \in X \mid Tu = u \}$

the set of <u>fixed points</u>

<u>Th</u>

X complete metric space (CMS)

$T : X \to X$ a-contraction

$\Rightarrow \exists^1 x^* \in F(T)$:

$\forall x \in X,\ T^n x \to x^*$

⟨Banach contraction principle⟩

$\begin{cases} x_1 \in C \\ x_n = T x_{n-1} \ (= T^{n-1} x_1) \end{cases}$

$\Rightarrow x_n \to x^*$

⟨Picard iteration⟩

Def.

X MS

$T: X \to X$ <u>Kannan mapping</u>

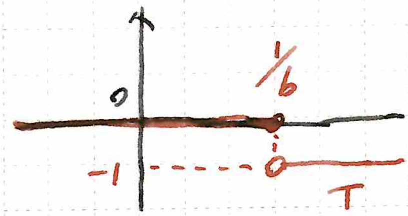
$\Leftrightarrow \exists b \in \left(0, \frac{1}{2}\right) : \forall x, y \in X,$

$$d(Tx, Ty) \leq b\big(d(x, Tx) + d(y, Ty)\big)$$

ex

Define $T : \mathbb{R} \to \mathbb{R}$ as follows:

$$Tx = \begin{cases} 0 & x \leq 1/b \\ -1 & x > 1/b \end{cases}$$



where $b \in \left(0, \frac{1}{2}\right)$.

Then, T is a Kannan mapping.

i.e $|Tx - Ty| \leq b\left(|x - Tx| + |y - Ty|\right)$.
$\forall x, y \in \mathbb{R}$.

(i) $x, y \leq 1/b$ or $x, y > 1/b$

As LHS $= |Tx - Ty| = 0$, OK.

(ii) Assume, w.l.g, that $x \leq \frac{1}{b} < y$.

Then, $Tx = 0$ and $Ty = -1$.

$\therefore$ LHS $= 1$.

On the other hand,

$$
\begin{aligned}
\text{RHS} &= b\left(|x| + |y+1|\right) \\
&\geq b|y+1| \\
&> b\left(\frac{1}{b} + 1\right) \\
&= 1 + b > 1.
\end{aligned}
$$

//

Th

X CMS
$T: X \to X$ b-Kannan mapping
i.e. $\exists b \in (0, \frac{1}{2}) : \forall x, y \in X$
$$d(Tx, Ty) \leq b\left(d(x, Tx) + d(y, Ty)\right)$$
$\Rightarrow \exists^1 x^* \in F(T) : \forall x \in X,\ T^n x \to x^*$

Proof.

⟨Existence⟩

Let $x \in X$ and define $x_n = T^n x$ $(n \in \mathbb{N} \cup \{0\})$.
We prove that (x_n) is a Cauchy sequence.

As T is b-Kannan,
$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq b\left(d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)\right) \\ &= b\left(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right) \end{aligned}$$

$\therefore (1-b)\, d(x_n, x_{n+1}) \leq b\, d(x_{n-1}, x_n)$
$\therefore d(x_n, x_{n+1}) \leq \frac{b}{1-b} d(x_{n-1}, x_n)$

Defining $\delta \equiv \frac{b}{1-b} \in (0, 1)$, we have
$$d(x_n, x_{n+1}) \leq \delta\, d(x_{n-1}, x_n). \quad \text{— (*)}$$
$\forall n \in \mathbb{N}$.

It holds that

$$
\begin{aligned}
d(x_n, x_{n+1}) &\leqq \delta d(x_{n-1}, x_n) \\
&\leqq \delta^2 d(x_{n-2}, x_{n-1}) \\
&\leqq \cdots \\
&\leqq \delta^n d(x_0, x_1).
\end{aligned}
$$

Let $m, n \in \mathbb{N} : m > n$.

We obtain

$$
\begin{aligned}
& d(x_n, x_m) \\
&\leqq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \\
&\leqq \delta^n d(x_0, x_1) + \cdots + \delta^{m-1} d(x_0, x_1) \\
&\leqq \delta^n d(x_0, x_1)(1 + \delta + \delta^2 + \cdots) \\
&= \frac{\delta^n}{1-\delta} d(x_0, x_1) \to 0 \quad (m, n \to \infty).
\end{aligned}
$$

This indicates that $\{x_n\} (\subset X)$ is a Cauchy sequence. ┘

As X is complete, $\exists x^* \in X : x_n \to x^*$.

$x^* = Tx^*$

It holds that

$$
\begin{aligned}
d(x^*, Tx^*) &\leqq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) \\
&= d(x^*, x_{n+1}) + d(Tx_n, Tx^*) \\
&\leqq d(x^*, x_{n+1}) \\
&\qquad + b\big(d(x_n, Tx_n) + d(x^*, Tx^*)\big) \\
&= d(x^*, x_{n+1}) \\
&\qquad + b\big(d(x_n, x_{n+1}) + d(x^*, Tx^*)\big).
\end{aligned}
$$

$$
\therefore (1-b)\, d(x^*, Tx^*) \leqq d(x^*, x_{n+1}) + b \cdot d(x_n, x_{n+1}).
$$

We obtain

$$
d(x^*, Tx^*) \leqq \frac{1}{1-b} d(x^*, x_{n+1}) + \frac{b}{1-b} d(x_n, x_{n+1}) \longrightarrow 0.
$$

$\therefore 0 \leqq d(x^*, Tx^*) \leqq 0.$

$\therefore x^* = Tx^*.$ ⌟

<Uniqueness>

Let $x^*, y^* \in F(T)$.

As T is b-Kannan,

$$\underbrace{d(Tx^*, Ty^*)}_{= d(x^*, y^*)} \le b\big(\underbrace{d(x^*, Tx^*) + d(y^*, Ty^*)}_{= 0}\big)$$

Therefore, $d(x^*, y^*) = 0$.

$\therefore x^* = y^*$.

X MS

$T: X \to X$

Consider the following condition:

$$\boxed{\begin{aligned} &\exists \delta \in (0,1) : \forall x, y \in X, \\ &d(Tx, Ty) \\ &\leq \delta \cdot \max\{d(x,y), d(x,Tx), d(y,Ty)\} \end{aligned}}$$

(*)

- $T: X \to X$ a-contraction
$\Rightarrow T$ satisfies (*) with $\delta = a$.

- $T: X \to X$ b-Kannan
$\Rightarrow$ T satisfies (*) with $\delta = 2b$.

($\because$)

As T is b-Kannan,

$$d(Tx, Ty) \leqq b\left(d(x, Tx) + d(y, Ty)\right)$$
$$= 2b \cdot \frac{d(x, Tx) + d(y, Ty)}{2}$$
$$\leqq 2b \cdot \max\{d(x, Tx), d(y, Ty)\}$$
$$\leqq 2b \cdot \max\{\underline{d(x, y)}, d(x, Tx), d(y, Ty)\}.$$

//

For $A, B \in \mathbb{R}$,

$A, B \geqq 0$のとき

$$\frac{A+B}{2} \leqq \max\{A, B\} \leqq A + B$$

Th

X CMS

$T: X \to X$

$\exists \delta \in (0,1): \forall x, y \in X,$

$d(Tx, Ty)$

$\leq \delta \cdot \max\{d(x,y), d(x,Tx), d(y,Ty)\}$

$\Rightarrow \exists 1\, x^* \in F(T): \forall x \in X, T^n x \to x^*$

Proof

Let $x \in X$ and

define $x_n = T^n x$ $(\forall n \in \mathbb{N} \cup \{0\})$.

We show that $\{x_n\}$ is a Cauchy sequence.

If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$,

then $x_{n+2} = Tx_{n+1} = Tx_n = x_{n+1}$.

Therefore, $x_n = x_{n+1} = x_{n+2} = \cdots$,

which means that

$\{x_n\}$ is a Cauchy sequence.

Assume, w.l.g., that

$x_n \neq x_{n+1}$ $\forall n \in \mathbb{N} \cup \{0\}$. —(*)

It follows that

$d(x_n, x_{n+1})$

$= d(Tx_{n-1}, Tx_n)$

$\leq \delta \cdot \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}),$

$d(x_n, Tx_n)\}$

$= \delta \cdot \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n),$

$d(x_n, x_{n+1})\}$

$= \delta \cdot \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$. $-(**)$

Here, $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ $\forall n \in \mathbb{N} \cup \{0\}$.

Suppose by contradiction that

$d(x_n, x_{n+1}) \geq d(x_{n-1}, x_n)$ for some n.

Then, from $(**)$,

$d(x_n, x_{n+1}) \leq \delta d(x_n, x_{n+1})$.

From $(*)$, $d(x_n, x_{n+1}) > 0$.

As $\delta \in (0,1)$, this is a contradiction. $\lrcorner$

From $(**)$, we have

$d(x_n, x_{n+1}) \leq \delta d(x_{n-1}, x_n)$. $-(*3)$

Using (*3), we obtain

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \delta\, d(x_{n-1}, x_n) \\ &\leq \delta^2 d(x_{n-2}, x_{n-1}) \\ &\leq \cdots \\ &\leq \delta^n d(x_0, x_1). \end{aligned}$$

Let $m, n \in \mathbb{N} : m > n$.

Then, $d(x_n, x_m)$

$$\begin{aligned} &\leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \\ &\leq \delta^n d(x_0, x_1) + \cdots + \delta^{m-1} d(x_0, x_1) \\ &\leq \delta^n d(x_0, x_1)(1 + \delta + \delta^2 + \cdots) \\ &= \frac{\delta^n}{1-\delta} d(x_0, x_1) \longrightarrow 0. \end{aligned}$$

Hence, $\{x_n\} (\subset X)$ is a Cauchy sequence. $\lrcorner$

As X is complete, $\exists x^* \in X : x_n \to x^*$.

$\underline{x^* = Tx^*}$

It follows that

$d(x^*, Tx^*)$

$\leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)$

$= d(x^*, x_{n+1}) + d(Tx_n, Tx^*)$

$\leq d(x^*, x_{n+1})$

$\quad + \delta \cdot \max\{d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*)\}$

$\leq d(x^*, x_{n+1})$

$\quad + \delta(d(x_n, x^*) + d(x_n, x_{n+1}) + d(x^*, Tx^*))$

As $n \to \infty$, we obtain

$$d(x^*, Tx^*) \leq \delta d(x^*, Tx^*).$$

As $\delta \in (0,1)$, we have $d(x^*, Tx^*) = 0$.

<Uniqueness>

Let $x^*, y^* \in F(T)$.

Then, $d(x^*, y^*) = d(Tx^*, Ty^*)$

$\leq \delta \cdot \max\{d(x^*, y^*), \underbrace{d(x^*, Tx^*), d(y^*, Ty^*)}_{=0}\}$

$= \delta d(x^*, y^*)$.

Therefore, $d(x^*, y^*) = 0$. //

Cor

$$\boxed{\begin{array}{l} X \text{ CMS} \\ T: X \to X \\ \exists a \in (0,1),\ b \in (0, \tfrac{1}{2}) : \forall x, y \in X, \\ \text{(a) } d(Tx, Ty) \leqq a\, d(x, y) \quad \boxed{\text{or}} \\ \text{(b) } d(Tx, Ty) \leqq b\big(d(x, Tx) + d(y, Ty)\big) \\ \Rightarrow \exists^1 x^* \in F(T) : \forall x \in X,\ T^n x \to x^* \end{array}}$$

Proof.

Define $\delta \equiv \max\{a, 2b\} \in (0,1)$.

Let $x, y \in X$.

It is sufficient to prove that

$$\underline{d(Tx, Ty) \leqq \delta \cdot \max\{d(x,y), d(x, Tx), d(y, Ty)\}.}$$

If (a) holds for $x, y \in X$, then

$$\begin{aligned} d(Tx, Ty) &\leqq a\, d(x,y) \\ &\leqq \delta \cdot \max\{d(x,y), d(x,Tx), d(y,Ty)\}. \end{aligned}$$

If (b) is true, then

$$\begin{aligned} d(Tx, Ty) &\leqq b\big(d(x, Tx) + d(y, Ty)\big) \\ &= 2b \cdot \frac{d(x, Tx) + d(y, Ty)}{2} \\ &\leqq \delta \cdot \max\{d(x, Tx), d(y, Ty)\} \\ &\leqq \delta \cdot \max\{d(x,y), d(x, Tx), d(y, Ty)\}. \end{aligned}$$

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Cor

X CMS

$T: X \to X$ a-contraction

$\Rightarrow \exists 1\, x^* \in F(T) : \forall x \in X,\ T^n x \to x^*$

Cor

X CMS

$T: X \to X$ b-Kannan mapping

$\Rightarrow \exists 1\, x^* \in F(T) : \forall x \in X,\ T^n x \to x^*$

Fixed point theorems in complete metric spaces

1. Xを距離空間, $T: X \to X$を連続写像とする. このとき, Tの不動点の集合$F(T)$はXにおける閉集合である. このことを示せ.

2. Vをベクトル空間, $T: V \to V$を線型写像とする. このとき, Tの不動点の集合$F(T)$はVの部分空間である. このことを示せ.

3. 縮小写像とその不動点定理の証明を復習せよ.

4. Kannan写像の定義を述べ, 例を挙げよ.

5. Kannan写像について, 縮小写像の不動点定理と同じ結論を導け. その証明において, 写像の連続性を用いないことを確認せよ.

6. 問題5で扱ったKannan写像についての不動点定理を若干一般化した次の定理を証明せよ.

定理A. 完備距離空間X上で定義された写像$T: X \to X$が, 条件

$$\exists \rho, \alpha \in (0,1) \text{ such that } \forall x, y \in X,$$
$$d(Tx, Ty) \leq \rho\{\alpha d(x, Tx) + (1-\alpha) d(y, Ty)\}$$

を満たすとする. このとき, Tの不動点$x^* \in X$がただ一つ存在し, Xの任意の点xに対して, $\{T^n x\}$はx^*に収束する.

7. 次の定理Bを証明せよ.

定理B. 完備距離空間X上で定義された写像$T: X \to X$が, 条件

$$\exists \delta \in (0,1) \text{ such that } \forall x, y \in X,$$
$$d(Tx, Ty) \leq \delta \cdot \max\{d(x,y),\ d(x, Tx),\ d(y, Ty)\}$$

を満たすとする. このとき, Tの不動点$x^* \in X$がただ一つ存在し, Xの任意の点xに対して, $\{T^n x\}$はx^*に収束する.

8. 定理Bを用いて次の定理Cを証明せよ.

定理C. 完備距離空間X上で定義された写像$T: X \to X$が, 条件

$$\exists a \in (0,1),\ b \in \left(0, \frac{1}{2}\right) \text{ such that } \forall x, y \in X,$$

$$\text{(a)} \quad d(Tx, Ty) \leq a d(x,y) \text{ or}$$
$$\text{(b)} \quad d(Tx, Ty) \leq b\{d(x, Tx) + d(y, Ty)\}$$

を満たすとする. このとき, Tの不動点x^*がただ一つ存在し、Xの任意の点xに対して, $\{T^n x\}$はx^*に収束する.

9. 定理BおよびCから縮小写像の不動点定理とKannan写像の不動点定理が導出されることを確認せよ.