Banach contraction principle

(X, d) metric space (MS)

$\Leftrightarrow$ (d1) $d(x,y) \geqq 0$; $d(x,y) = 0 \Leftrightarrow x = y$

(d2) $d(x,y) = d(y,x)$

(d3) $d(x,y) \leqq d(x,z) + d(z,y)$

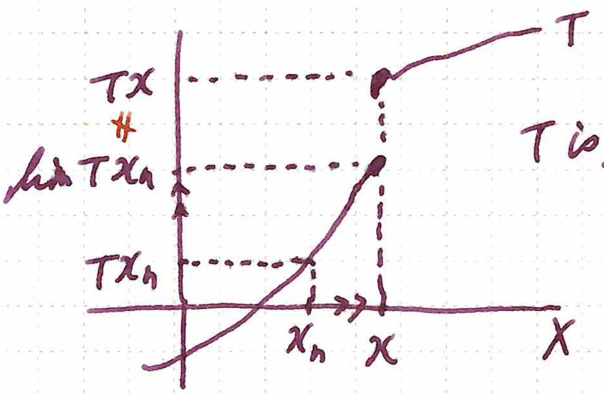
X, Y MSs (metric spaces)

- $T: X \to Y$ <u>continuous at $x \in X$</u>

$\Leftrightarrow$ $x_n \to x \Rightarrow Tx_n \to Tx$

- T: <u>continuous on X</u>.

$\Leftrightarrow$ $^{\forall} x \in X$, T is continuous at x.



T is <u>not</u> continuous at x.

$\exists \{x_n\} \subset X : \begin{cases} x_n \to x \\ Tx_n \not\to Tx \end{cases}$

$\{x_n\} \subset X$ <u>Cauchy sequence</u>

$\Leftrightarrow d(x_m, x_n) \to 0$ as $m, n \to \infty$

$\Leftrightarrow {}^\forall \varepsilon > 0, {}^\exists n_0 \in \mathbb{N}:$

$m, n \geqq n_0 \Rightarrow d(x_m, x_n) < \varepsilon$

$\{x_n\} \subset X$: <u>convergent</u> (in X).

$\Leftrightarrow {}^\exists x \in X : x_n \to x$

$\Leftrightarrow {}^\exists x \in X : {}^\forall \varepsilon > 0, {}^\exists n_0 \in \mathbb{N}:$

$n \geqq n_0 \Rightarrow d(x_n, x) < \varepsilon$.

X <u>complete metric space</u> (CMS)

$\Leftrightarrow$ (I) X : MS

(II) $\{x_n\} \subset X$: Cauchy sequence

$\Rightarrow \{x_n\}$: convergent

i.e. ${}^\exists x \in X : x_n \to x$

Lipschitz continuous mappings

Def.

X, Y MSs

$T: X \to Y$ <u>K-Lipschitz</u>

$\Leftrightarrow \exists K \geqq 0 : \forall x, y \in X,$

$$d(Tx, Ty) \leqq K d(x, y)$$

- $K \in [0, 1)$

  $\Leftrightarrow T$: contraction mapping 縮小写像

- $K = 1$

  $\Leftrightarrow T$: nonexpansive mapping (NE)

  非拡大写像

X, Y MSs

$T: X \to Y$ Lipschitz

i.e. $\exists K \geqq 0 : \forall x, y \in X,$

$$d(Tx, Ty) \leqq K d(x, y)$$

$\Rightarrow T$: continuous

Proof

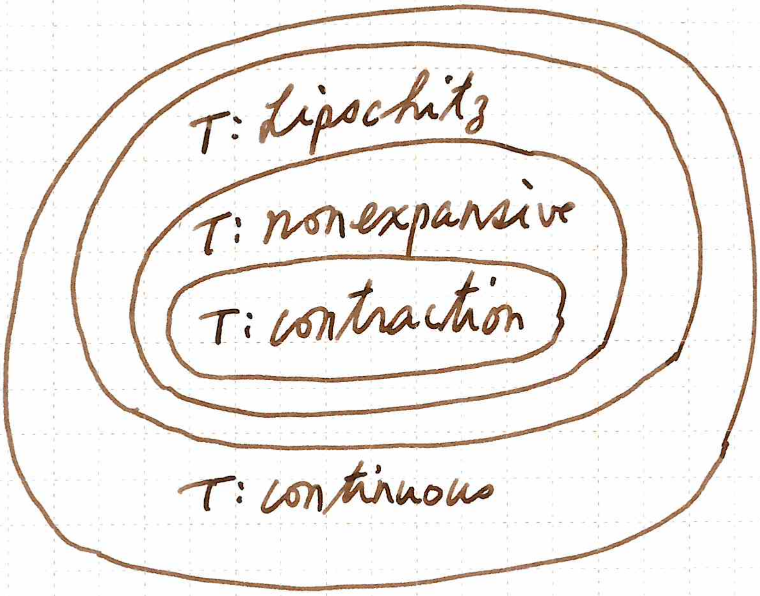
Let $x \in X$ and $\{x_n\} \subset X : x_n \to x$.

We prove that $\underline{Tx_n \to Tx}$.

It follows that

$$d(Tx_n, Tx) \leqq K d(x_n, x) \to 0.$$

$\therefore Tx_n \to Tx$. //



$I \subset \mathbb{R}$ open interval
$T: I \to \mathbb{R}$ differentiable
$\Rightarrow$ Equivalent
① T: K-Lipschitz
② $|T'(x)| \leq K \quad {}^{\forall} x \in I$

Proof

①$\Rightarrow$②

Let $x, y \in I : x \neq y$.
From ①, $|Tx - Ty| \leq K|x-y|$.

$$\therefore \left| \frac{Ty - Tx}{y - x} \right| \leq K$$

As $y \to x$, we have $|T'(x)| \leq K \quad {}^{\forall} x \in I$.

②$\Rightarrow$①

Let $x, y \in I$.
We show that $|Tx - Ty| \leq K|x-y|$.
Assume, w.l.g., that $x < y$.
From the mean value theorem,

$$^{\exists} c \in (x, y) : T'(c) = \frac{Tx - Ty}{x - y}.$$

$$\begin{aligned} \therefore |Tx - Ty| &= |T'(c)|\,|x-y| \\ &\leq K|x-y| \quad ② \end{aligned}$$

//

ex

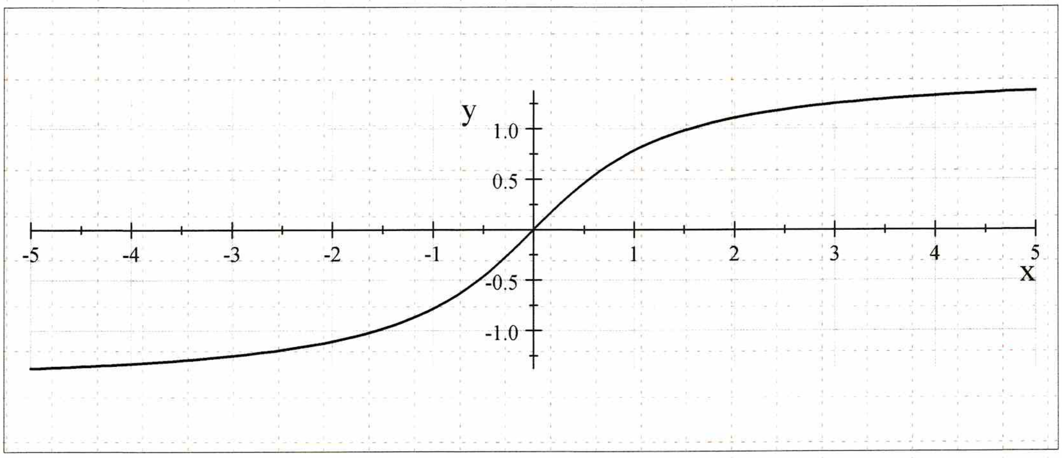
$T: \mathbb{R} \to \mathbb{R}$ NE (nonexpansive)

if $|Tx - Ty| \leq |x-y| \quad \forall x, y \in \mathbb{R}$

- $Tx = \sin x$
- $Tx = \cos x$
- $Tx = \mathrm{Tan}^{-1} x$

$\hookrightarrow T'(x) = \dfrac{1}{1+x^2} \in [0,1]$

$y = \mathrm{Tan}^{-1} x$



Fixed points

$X \neq \emptyset$

$C \subset X, \neq \emptyset$

$T : C \to X$

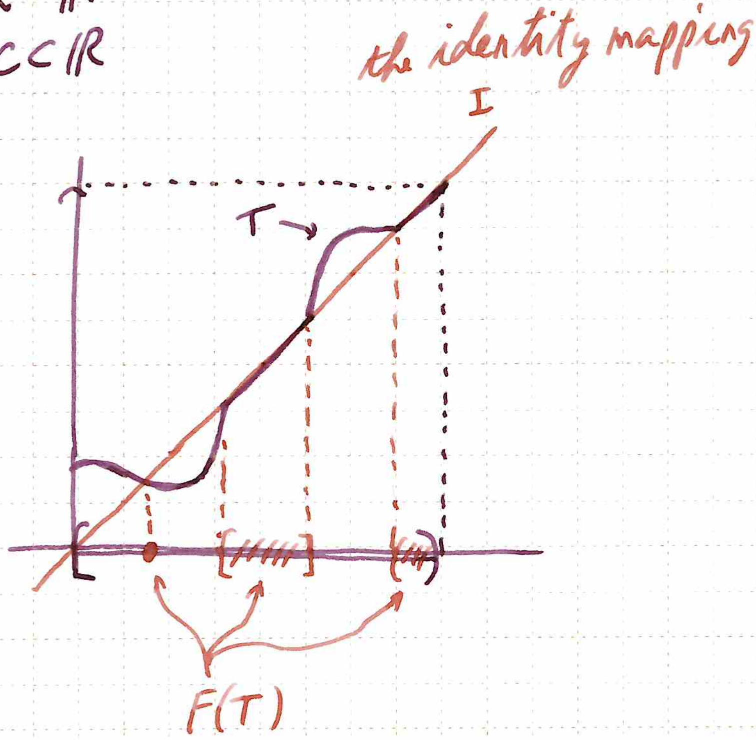
Then, $F(T) \equiv \{x \in C \mid Tx = x\}$.

the set of fixed points.

ex

$X = \mathbb{R}$

$C \subset \mathbb{R}$



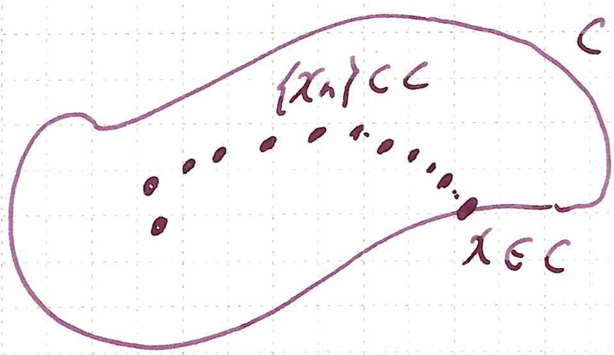
Def

X MS

$C \subset X$ <u>closed in X.</u>

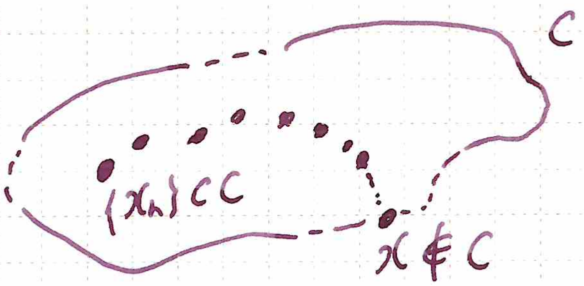
$\Longleftrightarrow \{x_n\} \subset C : x_n \to x \in X$

$\Longrightarrow x \in C$



※ $C \subset X$ is not closed in X.

$\Longleftrightarrow \exists \{x_n\} \subset C : \begin{cases} x_n \to x \\ x \notin C. \end{cases}$



X MS
$C \subset X, \neq \emptyset$
$T : C \to X$ continuous
$\Rightarrow F(T)$ is closed in C.

Proof.

Let $\{x_n\} \subset F(T) : x_n \to x \in C.$ — ①

i.e. $\forall n \in \mathbb{N},\ x_n = Tx_n$ — ②

We show that $x \in F(T)$.

i.e. $x = Tx$

As T is continuous and $x_n \to x$,
we have $Tx_n \to Tx$. — ③

Thus, the following hold:

$$d(x, Tx) \leq \underbrace{d(x, x_n)}_{\to 0 \text{ ①}} + \underbrace{d(x_n, Tx_n)}_{= 0 \text{ ②}} + \underbrace{d(Tx_n, Tx)}_{\to 0 \text{ ③}}$$

$\longrightarrow 0$ as $n \to \infty$.

We obtain $d(x, Tx) \leq 0$.

$\therefore d(x, Tx) = 0$.

$\therefore x = Tx$. //

X MS
$T: X \to X$ continuous
$x \in X$, $x_n = T^n x$ $(n \in \mathbb{N} \cup \{0\})$
$x_n \to x^*$
$\Rightarrow x^* \in F(T)$

Proof.

We prove that $\underline{x^* = Tx^*}$

It follows that

$$
\begin{aligned}
Tx^* &= T(\lim x_n) \\
&= \lim Tx_n \quad \text{T: continuous} \\
&= \lim x_{n+1} \\
&= x^*.
\end{aligned}
$$

//

Th

X CMS

$T: X \to X$ r-contraction

i.e. $\exists r \in (0,1): \forall x, y \in X,$

$$d(Tx, Ty) \leq r d(x, y)$$

$\Rightarrow \exists^1 x^* \in F(T): \forall x \in X, T^n x \to x^*$

Proof.

<Existence>

Let $x \in X$ and define $x_n \equiv T^n x$ $(n \in \mathbb{N} \cup \{0\})$.

$\{x_n\} (\subset X)$ is a Cauchy sequence.

For $n \in \mathbb{N}$, it holds that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq r d(x_{n-1}, x_n) \\ &\leq r^2 d(x_{n-2}, x_{n-1}) \leq \cdots \\ &\leq r^n d(x_0, x_1). \end{aligned}$$

Let $m, n \in \mathbb{N}: m \geq n$.

We have

$$\begin{aligned} &d(x_n, x_m) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq r^n d(x_0, x_1) + r^{n+1} d(x_0, x_1) + \cdots + r^{m-1} d(x_0, x_1) \\ &\leq r^n d(x_0, x_1) \cdot (1 + r + r^2 + \cdots\cdots) \\ &= \frac{1}{1-r} \cdot r^n d(x_0, x_1) \to 0 \quad (m, n \to \infty). \end{aligned}$$

As X is complete, $\exists x^* \in X : x_n \to x^*$.
We demonstrate that $\underline{x^* \in F(T)}$.

Indeed, $$\begin{aligned} Tx^* &= T(\lim x_n) \\ &= \lim Tx_n \\ &= \lim x_{n+1} = x^*. \end{aligned}$$ T: continuous

$\therefore Tx^* = x^*$.

<Uniqueness>

Let $x^*, y^* \in F(T)$.
We show that $\underline{x^* = y^*}$.
It follows that
$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\leqq r\, d(x^*, y^*). \end{aligned}$$
We have $(1-r)d(x^*, y^*) \leqq 0$.
As $r \in (0,1)$, $1-r > 0$.
Thus, dividing by $1-r\ (>0)$, we obtain
$$d(x^*, y^*) \leqq 0.$$
$\therefore x^* = y^*$.

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Th

X CMS
$T: X \to X$
$\exists M \in \mathbb{N}: T^M$; r-contraction
$\Rightarrow \exists^1 x^* \in F(T): {}^\forall x \in X, T^n x \to x^*$

Proof.
It holds that

$\exists^1 x^* \in F(T^M): {}^\forall x \in X, T^{M \cdot n} x \to x^* \ (n \to \infty)$. (*)

$F(T^M) = F(T)$

($\supset$) OK

($\subset$) Let $u \in F(T^M)$. i.e. $u = T^M u$.

Our aim is to prove that $u = Tu$.

It follows that

$$\begin{aligned} &d(u, Tu) \\ &\quad = d(T^M u, T^{M+1} u) \\ &\quad \leq r\, d(u, Tu). \end{aligned}$$

Therefore, $(1-r)\, d(u, Tu) \leq 0$.

As $1-r > 0$, $d(u, Tu) \leq 0$.

$\therefore u = Tu$. ⌟

From (*), $F(T) \left(= F(T^M)\right) = \{x^*\}$.

Let $x \in X$.

We demonstrate that $\underline{T^n x \to x^*.}$

From (*),

$$\begin{tikzcd}
x \arrow[d] & T^M x \arrow[d] & T^{2M} x \arrow[d] & \cdots \arrow[r] & x^* \\
Tx \arrow[d] & T^{M+1} x \arrow[d] & T^{2M+1} x \arrow[d] & \cdots \arrow[r] & x^* \\
T^2 x \arrow[d] & T^{M+2} x \arrow[d] & T^{2M+2} x \arrow[d] & \cdots \arrow[r] & x^* \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & & \\
T^{M-1} x \arrow[uuuur, bend right] & T^{2M-1} x \arrow[uuuur, bend right] & T^{3M-1} x & \cdots \arrow[r] & x^*
\end{tikzcd}$$

This indicates that $T^n x \to x^*$. //

Let $x \in X$

Formal proof of $T^n x \to x^*$ $(M=3)$

From (*), $T^{3n} u \to x^*$ $\forall u \in X$.

Let $\varepsilon > 0$.

For $u = x \in X$,

$\exists n_1 \in \mathbb{N} : n \geq n_1 \Rightarrow d(T^{3n} x, x^*) < \varepsilon$.

For $u = Tx \in X$,

$\exists n_2 \in \mathbb{N} : n \geq n_2 \Rightarrow d(T^{3n}(Tx), x^*) < \varepsilon$.

For $u = T^2 x \in X$,

$\exists n_3 \in \mathbb{N} : n \geq n_3 \Rightarrow d(T^{3n}(T^2 x), x^*) < \varepsilon$.

Define

$n_0 = \max\{3n_1, 3n_2+1, 3n_3+2\} \in \mathbb{N}$.

Let $n \geq n_0$.

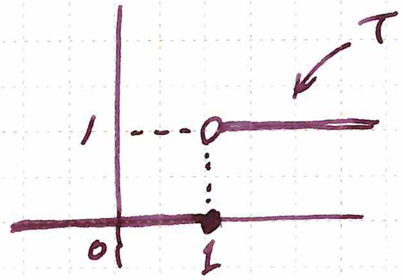
Then, $d(T^n x, x^*) < \varepsilon$.

$\therefore \forall x \in X$, $T^n x \to x^*$. //

ex

$X = \mathbb{R}$

$$Tx = \begin{cases} 1 & x > 1 \\ 0 & x \leq 1 \end{cases}$$



- T is not continuous and therefore, it is not a contraction mapping.
- However, $T^2 = 0$ and it is a contraction mapping.

The Banach contraction principle

1. Let X be a metric space and let $\{x_n\}$ be a sequence in X that is convergent. Show that $\{x_n\}$ is a Cauchy sequence.

2. Let X, Y be metric spaces and let $T : X \to Y$ be a Lipschitz continuous mapping. Show that T is continuous.

3. Let I be an open interval in $\mathbb{R}$ and let $T : I \to \mathbb{R}$ be a differentiable function. Prove that the following two assertions (1) and (2) are equivalent:
(1) T is K-Lipschitz continuous.
(2) $|T'(x)| \leq K$ for all $x \in I$, where T' is the derivative of T.

4. Let X be a metric space, let C be a nonempty subset of X, and let $T : C \to X$ be a continuous mapping. Then, the set of fixed points of T

$$F(T) = \{x \in C : Tx = x\}$$

is closed in C. Prove this.

5. Let X be a metric space and let $T : X \to X$ be an r-contraction mapping. For $x \in X$, define $x_n = T^n x$ for $n \in \mathbb{N} \cup \{0\}$ and suppose that $x_n \to x^*$ for some $x^* \in X$. Prove that $x^* \in F(T)$.

6. Write the statement of the Banach contraction principle and prove it.

7. Let X be a complete metric space. Assume that there exists $M \in \mathbb{N}$ such that $T^M : X \to X$ is an r-contraction mapping, where T^M is the M-time composite mapping of T. In this case, the same conclusion as the Banach contraction principle holds. Prove this.

8. Give an example of a mapping T where T and T^2 are not contraction mappings, and T^3 is a contraction mapping.

Reference

S. Banach, "Sur les opéations dans les ensembles abstraits et leur applications aux éuations intérales," Fund. Math. 3 (1922): 133-181.