Some results on the Banach contraction principle

X MS

$x, y, z \in X$

$\Rightarrow |d(x,y) - d(x,z)| \leq d(y,z)$

Proof.

It is sufficient to show that

$$-d(y,z) \overset{①}{\leq} d(x,y) - d(x,z) \overset{②}{\leq} d(y,z).$$

①: It follows that

$$d(x,z) \leq d(x,y) + d(y,z).$$

From this,

$$-d(y,z) \leq d(x,y) - d(x,z). \quad \lrcorner$$

②: The following expression holds:

$$d(x,y) \leq d(x,z) + d(z,y).$$

$$\therefore d(x,y) - d(x,z) \leq d(z,y) = d(y,z).$$

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$x_n \to x$
$y_n \to y$
$\Rightarrow d(x_n, y_n) \to d(x, y)$

Proof

We have

$$|d(x_n, y_n) - d(x, y)|$$
$$\leq |d(x_n, y_n) - d(x_n, y)| + |d(x_n, y) - d(x, y)|$$
$$\leq d(y_n, y) + d(x_n, x)$$
$$\to 0.$$

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X MS

$M \subset X$

Def

$x \in X$ is a <u>contact point</u> of M.

$\Leftrightarrow \forall \varepsilon > 0,\ S_\varepsilon(x) \cap M \neq \emptyset$

$\overline{M} = \{x \in X \mid x \text{ is a contact point of } M\}$

← the <u>closure</u> of M

X MS
$M \subset X$
$\Rightarrow \overline{M} = \{x \in X \mid \exists \{x_n\} \subset M : x_n \to x\}$

Proof.

We show that the following assertions are equivalent:

(i) $x \in \overline{M}$ i.e. $\forall \varepsilon > 0, S_\varepsilon(x) \cap M \neq \emptyset$,

(ii) $\exists \{x_n\} \subset M : x_n \to x$.

(i) $\Rightarrow$ (ii)

Letting $\varepsilon = 1/n > 0$ in (i), we have

$\forall n \in \mathbb{N}, \exists x_n \in M : d(x_n, x) < \frac{1}{n}$.

$\therefore \exists \{x_n\} \subset M : x_n \to x$. ⌟

(ii) $\Rightarrow$ (i)

Let $\varepsilon > 0$.

From (ii), $\exists n_0 \in \mathbb{N} : n \geq n_0 \Rightarrow x_n \in S_\varepsilon(x)$.

As $x_{n_0} \in S_\varepsilon(x) \cap M$,

we obtain $S_\varepsilon(x) \cap M \neq \emptyset \ \forall \varepsilon > 0$. //

X, Y MSs
$T_n : X \to Y$ K_n-Lipschitz $(n \in \mathbb{N})$
$\{K_n\} (\subset [0, \infty))$: bdd
$T : X \to Y$
$\forall x \in X, \ T_n x \to Tx$ — (*)
$\Rightarrow T$: Lipschitz

<u>Proof.</u>

As $\{K_n\}$ is bdd,
$\exists \{K_{n_i}\} \subset \{K_n\}, K \in [0, \infty) : K_{n_i} \to K$. — (**)

Let $x, y \in X$.

We prove that <u>$d(Tx, Ty) \leq K d(x, y)$</u>.

It follows that

$$\begin{aligned} & d(Tx, Ty) \\ & = d\left(\lim_{i\to\infty} T_{n_i} x, \lim_{i\to\infty} T_{n_i} y\right) \quad (*) \\ & = \lim_{i\to\infty} d(T_{n_i} x, T_{n_i} y) \quad T_{n_i} : K_{n_i}\text{-Lipschitz} \\ & \leq \lim K_{n_i} d(x, y) \quad K_{n_i} \to K \ (**) \\ & = K d(x, y). \end{aligned}$$

//

X, Y MSs

$T_n : X \to Y$ K_n-Lipschitz $(n \in \mathbb{N})$

$K_n \to K \in [0, \infty)$

$T : X \to Y$

$\forall_{x \in X},\ T_n x \to Tx \ (n \to \infty)$

$\Rightarrow T$: K-Lipschitz

Proof.

Let $x, y \in X$.

We aim to show that

$$d(Tx, Ty) \leq K d(x, y).$$

This can be proved as follows:

$$
\begin{aligned}
& d(Tx, Ty) \\
&= d(\lim T_n x, \lim T_n y) \\
&= \lim d(T_n x, T_n y) \\
&\leq \lim K_n d(x, y) \quad \text{(} T_n : K_n\text{-Lipschitz)} \\
&= K d(x, y).
\end{aligned}
$$

//

Th

> X MS
> $T_n : X \to X$ r-contraction ($n \in \mathbb{N}$)
> with $\{x_n^*\} = F(T_n)$
> $T : X \to X$ with $x^* \in F(T)$
> $\forall x \in X,\ T_n x \to Tx \quad (n \to \infty)$
> $\Rightarrow x_n^* \to x^*$

Proof.

It follows that

$$\begin{aligned} d(x_n^*, x^*) &= d(T_n x_n^*, Tx^*) \\ &\leq d(T_n x_n^*, T_n x^*) + d(T_n x^*, Tx^*) \\ &\leq r\, d(x_n^*, x^*) + d(T_n x^*, Tx^*). \end{aligned}$$

Thus, $(1-r)\, d(x_n^*, x^*) \leq d(T_n x^*, Tx^*)$.

As $1-r > 0$,

$$d(x_n^*, x^*) \leq \frac{1}{1-r} d(T_n x^*, Tx^*) \to 0.$$

$\therefore x_n^* \to x^*$.

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Th

> X, Λ MSs
> $\forall \alpha \in \Lambda$, $T(\cdot, \alpha) : X \to X$ r-contraction
> with $x_\alpha^* = T(x_\alpha^*, \alpha)$
> $\forall x \in X$, $T(x, \cdot) : \Lambda \to X$ continuous.
> $\Rightarrow x_\alpha^* \to x_\beta^*$ (as $\alpha \to \beta$)

Proof.

It holds true that

$$
\begin{aligned}
& d(x_\alpha^*, x_\beta^*) \\
& = d(T(x_\alpha^*, \alpha), T(x_\beta^*, \beta)) \\
& \leq d(T(x_\alpha^*, \alpha), T(x_\beta^*, \alpha)) \\
& \qquad + d(T(x_\beta^*, \alpha), T(x_\beta^*, \beta)) \\
& \leq r\, d(x_\alpha^*, x_\beta^*) \\
& \qquad + d(T(x_\beta^*, \alpha), T(x_\beta^*, \beta))
\end{aligned}
$$

$$
\therefore d(x_\alpha^*, x_\beta^*) \leq \frac{1}{1-r} d(T(x_\beta^*, \alpha), T(x_\beta^*, \beta)) \to 0 \text{ as } \alpha \to \beta
$$

$\therefore x_\alpha^* \to x_\beta^*$ as $\alpha \to \beta$. //

Th

> X MS
>
> $T_n : X \to X$ K_n-Lipschitz $(n \in \mathbb{N})$
>
> $\{K_n\}$ $(\subset [0,\infty))$: bdd
>
> $x_n^* \in F(T_n)$
>
> $T : X \to X$
>
> $\forall_{x \in X},\ T_n x \to Tx$
>
> $x_n^* \to x^* \in X$
>
> $\Rightarrow x^* \in F(T)$

Proof

As $\{K_n\}$ is bdd,

$\exists \bar{K} > 0 : \forall_{n \in \mathbb{N}},\ 0 \leq K_n \leq \bar{K}$.

It follows that

$$
\begin{aligned}
d(x^*, Tx^*) &\leq d(x^*, x_n^*) + \underline{d(x_n^*, T_n x_n^*)}^{\,=0} \\
&\qquad + d(T_n x_n^*, T_n x^*) + d(T_n x^*, Tx^*) \\
&\leq d(x^*, x_n^*) + K_n d(x_n^*, x^*) + d(T_n x^*, Tx^*) \\
&\leq d(x^*, x_n^*) + \bar{K} d(x_n^*, x^*) + d(T_n x^*, Tx^*) \\
&\longrightarrow 0.
\end{aligned}
$$

$\therefore d(x^*, Tx^*) \leq 0$. $\therefore x^* = Tx^*$. //

Cor

X MS

$T_n : X \to X$ r_n-contraction

$r_n \to r \in [0,1)$

$\{x_n^*\} = F(T_n)$

$T : X \to X$

$\forall x \in X,\ T_n x \to Tx$

$x_n^* \to x^*$

$\Rightarrow x^* \in F(T)$

Th

> X MS
> $T: X \to X$ r-contraction
> $x \in X$, $x_n = T^n x$ $(n \in \mathbb{N} \cup \{0\})$
> $x_{n_i} \to x^* \in X$,
> where $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$.
> $\Rightarrow$ x^* is the unique fixed point of T.

Proof.

First, we verify that $x^* = Tx^*$.

It holds that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(x_n, Tx_n) \\ &= d(Tx_{n-1}, Tx_n) \\ &\leq r\, d(x_{n-1}, x_n) \\ &\leq \cdots \\ &\leq r^n d(x_0, x_1) \to 0 \text{ as } n \to \infty. \quad (*) \end{aligned}$$

Using $(*)$, we have

$$\begin{aligned} d(x^*, Tx^*) &= d(\lim x_{n_i}, T(\lim x_{n_i})) \\ &= d(\lim x_{n_i}, \lim Tx_{n_i}) \\ &= \lim d(x_{n_i}, Tx_{n_i}) \quad (*) \\ &= 0. \end{aligned}$$

$\therefore x^* = Tx^*$. ⌟

Uniqueness

Let $x^*, y^* \in F(T)$.

It follows that

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq r\,d(x^*, y^*).$$

$\therefore (1-r)\,d(x^*, y^*) \leq 0.$

As $1-r > 0$, we have $d(x^*, y^*) \leq 0$.

$\therefore x^* = y^*.$

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Lemma

> X, Y MSs
> $M \subset X : \overline{M} = X$
> $T : X \to X$ continuous
> K-Lipschitz on M
> $\Rightarrow T$: K-Lipschitz

Proof.

Let $x, y \in X \,(= \overline{M})$.

We show that $d(Tx, Ty) \leq K d(x, y)$.

As $x, y \in \overline{M}$, $\begin{cases} \exists \{x_n\} \subset M : x_n \to x \\ \exists \{y_n\} \subset M : y_n \to y. \end{cases}$ — (*)

As T is continuous,

$Tx_n \to Tx, \quad Ty_n \to Ty.$ — (**)

It follows that

$$\begin{aligned} d(Tx, Ty) &= d(\lim Tx_n, \lim Ty_n) \quad \leftarrow (**) \\ &= \lim d(Tx_n, Ty_n) \\ &\leq \lim K d(x_n, y_n) \quad \text{(} x_n, y_n \in M,\ T\text{: K-Lipschitz on } M \text{)} \\ &= K d(\lim x_n, \lim y_n) \\ &= K d(x, y). \quad (*) \end{aligned}$$

//

Th

X MS

$M \subset X : \overline{M} = X$

$T : X \to X$ continuous

r-contraction on M

$x \in X$, $x_n = T^n x$ $(n \in \mathbb{N} \cup \{0\})$

$x_{n_i} \to x^*$

$\Rightarrow x^*$ is the unique fixed point of T.

Proof

From Lemma,

T is r-contraction on X.

From the previous theorem,

we obtain the desired result. //

Th

> X MS
> $T: X \to X$ r-contraction
> $\Rightarrow F(T^M) = F(T) \ \forall M \in \mathbb{N}$

Proof.

($\supset$) OK.

($\subset$) Let $u \in F(T^M)$.

i.e. $u = T^M u$

We prove that $\underline{u \in F(T)}$.

i.e. $u = Tu$.

It follows that

$d(u, Tu)$

$= d(T^M u, T^{M+1} u)$

$\leqq r d(T^{M-1} u, T^M u)$

$\cdots$

$\leqq r^M d(u, Tu)$.

$\therefore (1 - r^M) d(u, Tu) \leqq 0$.

We obtain $u = Tu$. //

※ T has no nontrivial periodic points.

$u \in X$ <u>M-periodic point</u> (nontrivial)

$$\Leftrightarrow \begin{cases} u \in F(T^M) \\ u \notin F(T^m) \quad m = 1, \cdots, M-1 \end{cases}$$

Some Results on the Banach contraction principle

1. Let $x, y, z \in X$, where X is a metric space. Prove that $|d(x,y) - d(x,z)| \leq d(y,z)$.

2. Let $\{x_n\}$ and $\{y_n\}$ be sequences in a metric space X such that $x_n \to x$ and $y_n \to y$. Show that $d(x_n, y_n) \to d(x,y)$.

3. Let M be a subset of a metric space X. The *closure* of M is usually defined as

$$\overline{M} = \{x \in X : \forall \varepsilon > 0,\ S_\varepsilon(x) \cap M \neq \emptyset\},$$

where $S_\varepsilon(x)$ is the open sphere of radius ε centered on x. Show that

$$\overline{M} = \{x \in X : \exists \{x_n\} \subset M \text{ such that } x_n \to x\}.$$

4. Let X and Y be metric spaces. Let $T_n : X \to Y$ be a K_n–Lipschitz continuous mapping ($\forall n \in \mathbb{N}$). Assume that the sequence of Lipschitz constants $\{K_n\}$ is bounded and that $T_n x \to Tx$ for all $x \in X$, where $T : X \to Y$. Prove that T is a Lipschitz continuous mapping.

5. Let $T_n : X \to X$ be an r–contraction mapping with $\{x_n^*\} = F(T_n)$ ($\forall n \in \mathbb{N}$), where X is a metric space. Let T be a self-mapping defined on X with $x^* \in F(T)$. Assume that $T_n x \to Tx$ for all $x \in X$. Prove that $x_n^* \to x^*$. (See Theorem 1.2. in [1])

6. Let X and Λ be metric spaces. Let $T : X \times \Lambda \to X$ such that $T(\cdot, \alpha) : X \to X$ is an r–contraction mapping with $\{x_\alpha^*\} = T(x_\alpha^*, \alpha)$ for all $\alpha \in \Lambda$. Suppose that $T(x, \cdot) : \Lambda \to X$ is continuous for all $x \in X$. Show that $x_\alpha^* \to x_\beta^*$ as $\alpha \to \beta$.

7. Let $T_n : X \to X$ be a K_n–Lipschitz continuous mapping with $x_n^* \in F(T_n)$ ($\forall n \in \mathbb{N}$), where X is a metric space. Assume that $\{K_n\}$ is bounded. Let T be a self-mapping defined on X. Assume that $T_n x \to Tx$ for all $x \in X$. Show the following: If $x_n^* \to x^*$, then $x^* \in F(T)$.

8. Let X be a metric space and let $T : X \to X$ be an r–contraction mapping. For $x \in X$, define $x_n = T^n x$ for $n \in \mathbb{N} \cup \{0\}$. Assume that $x_{n_i} \to x^*$ for some $x^* \in X$, where $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$. Prove that $x^* \in F(T)$.

9. Let X and Y be metric spaces and let $M \subset X$ such that $\overline{M} = X$. Let $T : X \to Y$ be a continuous mapping. Assume that T is a K–Lipschitz continuous mapping on M. Show that T is a K–Lipschitz continuous mapping (on X).

10. Let X be a metric space and let $M \subset X$ such that $\overline{M} = X$. Let $T : X \to Y$ be a continuous mapping. Assume that T is an r–contraction mapping on M. For $x \in X$, define $x_n = T^n x$ for $n \in \mathbb{N} \cup \{0\}$. Assume that $x_{n_i} \to x^*$ for some $x^* \in X$, where $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$. Prove that x^* is a unique element of $F(T)$.

11. Let X be a metric space and let $T : X \to X$ be an r–contraction mapping. Show that T has no nontrivial periodic points.

References

[1] Frank F. Bonsall, "Lectures on some fixed point theorems of functional analysis," No. 26. Bombay: Tata Institute of Fundamental Research, (1962).

[2] R. Kannan, "Some Results on Fixed Points-II," The American Mathematical Monthly, 76(4) (1969): 405-08.