Contraction-type mappings

Def.

X, Y MSs

$T : X \to Y$ <u>a-contraction</u>

$\Leftrightarrow \exists a \in [0,1) : \forall x, y \in X,$

$d(Tx, Ty) \leq a\, d(x,y)$

$\Leftrightarrow \exists a \in (0,1) : \forall x, y \in X,$

$d(Tx, Ty) \leq a\, d(x,y).$

Def.

X MS

$T: X \to X$ <u>Kannan mapping</u>

$\Leftrightarrow \exists b \in (0, \frac{1}{2}) : \forall x, y \in X,$

$$d(Tx, Ty) \leq b(d(x, Tx) + d(y, Ty))$$

$\Leftrightarrow T$: b-Kannan

※ $T: X \to X$ is required.

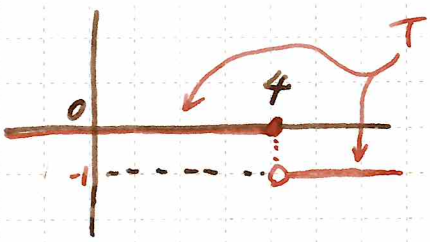
ex 1

$X = \mathbb{R}$

$T: X \to X$ defined as follows:

$$Tx = \begin{cases} 0 & x \leq 4 \\ -1 & x > 4 \end{cases}$$

$\Rightarrow T$ is a Kannan mapping.



We show that

$$|Tx - Ty| \leq \frac{1}{5}\left(|x - Tx| + |y - Ty|\right). \quad -(*)$$

(i) $x, y \leq 4$ or $x, y > 4$ OK

(ii) $x \leq 4 < y$

LHS of $(*) = |Tx - Ty| = |0 - (-1)| = 1.$

$$\begin{aligned} \text{RHS} &= \frac{1}{5}\left(|x - Tx| + |y - Ty|\right) \\ &\geq \frac{1}{5}|y - Ty| \\ &= \frac{1}{5}|y+1| \geq \frac{1}{5}|4+1| = 1 = \text{LHS}. \end{aligned}$$

//

ex 2

$X = [0,1]\ (\subset \mathbb{R})$

$T : X \to X$ defined as follows

$$Tx = \frac{1}{3}x \quad \forall x \in X$$

$\Rightarrow T$ is *not* a Kannan mapping.

i.e. $\forall b \in \left(0, \frac{1}{2}\right), \exists x, y \in X$:

$$d(Tx, Ty) > b\big(d(x, Tx) + d(y, Ty)\big) \quad -(*)$$

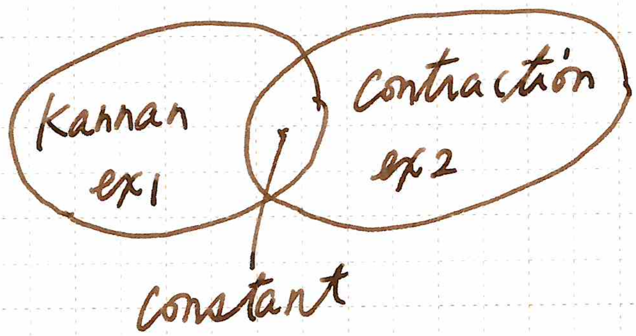
Let $x = 1/3,\ y = 0 \in X$.

Then, LHS of $(*) = 1/9$.

On the other hand,

$$RHS = b\,|x - Tx| = b\left|\frac{1}{3} - \frac{1}{9}\right|$$

$$= \frac{2}{9}b < \frac{1}{9} = LHS. \quad //$$



Th

X CMS
$T: X \to X$ b-Kannan
$\Rightarrow \exists^1 x^* \in F(T) : \forall x \in X, T^n x \to x^*$

Proof.

<Existence>

Let $x \in X$ and define $x_n = T^n x$ $(n \in \mathbb{N} \cup \{0\})$.

We demonstrate that

$\{x_n\}$ is a Cauchy sequence.

It holds that

$$\begin{aligned} &d(x_n, x_{n+1}) \\ &= d(Tx_{n-1}, Tx_n) \\ &\leqq b\left(d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)\right) \\ &= b\left(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right) \end{aligned}$$

$\therefore (1-b)\, d(x_n, x_{n+1}) \leqq b \cdot d(x_{n-1}, x_n)$.

$\therefore d(x_n, x_{n+1}) \leqq \frac{b}{1-b} d(x_{n-1}, x_n)$. $-(*)$

Define $\delta \equiv \frac{b}{1-b}$.

As $b \in (0, \frac{1}{2})$, we have $\delta \in (0, 1)$.

From (*),

$$d(x_n, x_{n+1}) \leqq \delta d(x_{n-1}, x_n)$$
$$\leqq \delta^2 d(x_{n-2}, x_{n-1})$$
$$\leqq \cdots \leqq \delta^n d(x_0, x_1).$$

Let $m, n \in \mathbb{N} : m \geqq n$.

The following expressions hold :

$$d(x_n, x_m)$$
$$\leqq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m)$$
$$\leqq \delta^n d(x_0, x_1) + \delta^{n+1} d(x_0, x_1) + \cdots + \delta^{m-1} d(x_0, x_1)$$
$$\leqq \delta^n d(x_0, x_1) (1 + \delta + \delta^2 + \cdots)$$
$$= \delta^n d(x_0, x_1) \cdot \frac{1}{1-\delta} \to 0 \quad m, n \to \infty$$

This indicates that
$\{x_n\}$ is a Cauchy sequence. ⌋

As X is complete, $\exists x^* \in X : x_n \to x^*$.

$x^* = Tx^*$

It holds that

$d(x^*, Tx^*)$

$\leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)$

$= d(x^*, x_{n+1}) + d(Tx_n, Tx^*)$

$\leq d(x^*, x_{n+1})$

$\quad + b\left(d(x_n, x_{n+1}) + d(x^*, Tx^*)\right)$

We have

$d(x^*, Tx^*)$

$\leq \frac{1}{1-b} d(x^*, x_{n+1}) + \frac{b}{1-b} d(x_n, x_{n+1})$

$\longrightarrow 0$ as $n \to \infty$.

$\therefore d(x^*, Tx^*) \leq 0$.

$\therefore x^* = Tx^*$. ⌟

⟨Uniqueness⟩

Let $x^*, y^* \in F(T)$.

It follows that

$d(x^*, y^*) = d(Tx^*, Ty^*)$

$\quad \leq b\left(d(x^*, Tx^*) + d(y^*, Ty^*)\right) = 0$

$\therefore x^* = y^*$. //

<u>Def</u>

X MS

$T: X \to X$ <u>Chatterjea mapping</u>

$\Leftrightarrow {}^{\exists}c \in \left(0, \frac{1}{2}\right) : {}^{\forall}x, y \in X,$

$$d(Tx, Ty) \leq c\big(d(x, Ty) + d(Tx, y)\big)$$

$\Leftrightarrow T:$ c-Chatterjea

Th

X CMS
$T: X \to X$ c-Chatterjea
$\Rightarrow \exists^1 x^* \in F(T): \forall x \in X,\ T^n x \to x^*$

Proof.

⟨Existence⟩

Let $x \in X$ and $x_n = T^n x$ $(n \in \mathbb{N} \cup \{0\})$.

<u>$\{x_n\}$ is a Cauchy sequence.</u>

We have the following:

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$
$$\leq c\left(d(x_{n-1}, Tx_n) + \underbrace{d(Tx_{n-1}, x_n)}_{=0}\right)$$
$$= c\, d(x_{n-1}, x_{n+1})$$
$$\leq c\, d(x_{n-1}, x_n) + c\, d(x_n, x_{n+1})$$

We obtain

$$d(x_n, x_{n+1}) \leq \frac{c}{1-c} d(x_{n-1}, x_n).$$

Define $\delta \equiv \frac{c}{1-c}$. Then, $\delta \in (0,1)$.

We have

$$d(x_n, x_{n+1}) \leqq \delta\, d(x_{n-1}, x_n)$$
$$\leqq \cdots \leqq \delta^n d(x_0, x_1).$$

Let $m, n \in \mathbb{N} : m \geqq n$.

It follows that

$$d(x_n, x_m)$$
$$\leqq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m)$$
$$\leqq \delta^n d(x_0, x_1) + \delta^{n+1} d(x_0, x_1) + \cdots$$
$$\cdots + \delta^{m-1} d(x_0, x_1)$$
$$\leqq \delta^n d(x_0, x_1)(1 + \delta + \delta^2 + \cdots)$$
$$= \delta^n d(x_0, x_1) \frac{1}{1-\delta} \to 0 \quad (m, n \to \infty).$$

This implies that

$\{x_n\}$ is a Cauchy sequence. ⌟

As X is complete, $\exists x^* \in X : x_n \to x^*$.

$x^* = Tx^*$

This can be verified as follows:

$d(x^*, Tx^*)$

$\leqq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)$

$= d(x^*, x_{n+1}) + d(Tx_n, Tx^*)$

$\leqq d(x^*, x_{n+1})$

$\quad + c\big(d(x_n, Tx^*) + d(x_{n+1}, x^*)\big).$

As $n \to \infty$, $d(x^*, Tx^*) \leqq c\, d(x^*, Tx^*)$.

$\therefore (1-c)\, d(x^*, Tx^*).$

As $1-c > 0$, $d(x^*, Tx^*) \leqq 0$.

$\therefore x^* = Tx^*.$

⟨Uniqueness⟩

Let $x^*, y^* \in F(T)$.

Then, $d(x^*, y^*) = d(Tx^*, Ty^*)$

$\leqq c\big(d(x^*, Ty^*) + d(Tx^*, y^*)\big).$

$= 2c\, d(x^*, y^*).$

$\therefore (1-2c)\, d(x^*, y^*) \leqq 0.$

As $1-2c > 0$, $d(x^*, y^*) \leqq 0$. $\therefore x^* = y^*$. //

Def.

$T: X \to X$ Zamfirescu mapping
(Z-mapping)

$\Longleftrightarrow \exists a \in (0,1),\ b, c \in (0, \frac{1}{2}) : \forall x, y \in X,$

(a) $d(Tx, Ty) \leq a\, d(x, y)$ $\boxed{\text{or}}$

(b) $d(Tx, Ty) \leq b\big(d(x, Tx) + d(y, Ty)\big)$ $\boxed{\text{or}}$

(c) $d(Tx, Ty) \leq c\big(d(x, Ty) + d(Tx, y)\big)$

$\downarrow$ special cases

(a): T: a-contraction
(b): T: b-Kannan
(c): T: c-Chatterjea

Th

X CMS

$T: X \to X$ Zamfirescu mapping

i.e. $\exists a \in (0,1),\ b, c \in (0, \frac{1}{2}) : \forall x, y \in X,$

(a) $d(Tx, Ty) \leq a\, d(x,y)$, $\boxed{\text{or}}$

(b) $d(Tx, Ty) \leq b\big(d(x, Tx) + d(y, Ty)\big)$, $\boxed{\text{or}}$

(c) $d(Tx, Ty) \leq c\big(d(x, Ty) + d(Tx, y)\big)$.

$\Rightarrow \exists^1 x^* \in F(T) : \forall x \in X,\ T^n x \to x^*$

Proof

⟨Existence⟩

Define $\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\} \in (0,1)$.

Let $x \in X$ and define $x_n = T^n x$ $(n \in \mathbb{N} \cup \{0\})$.

We prove that $\{x_n\}$ is a Cauchy sequence.

Observe that

$d(x_n, x_{n+1}) \leq \delta\, d(x_{n-1}, x_n)\ \forall n \in \mathbb{N}$.

(a) for (x_{n-1}, x_n)

We have

$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$

$\leq a\, d(x_{n-1}, x_n) \leq \delta\, d(x_{n-1}, x_n)$. ⌋

(b) for (x_{n-1}, x_n)

It follows that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$
$$\leq b\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big).$$

$$\therefore d(x_n, x_{n+1}) \leq \frac{b}{1-b} d(x_{n-1}, x_n)$$
$$\leq \delta d(x_{n-1}, x_n). \quad \lrcorner$$

(c) for (x_{n-1}, x_n)

We obtain

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$
$$\leq c\big(d(x_{n-1}, Tx_n) + d(Tx_{n-1}, x_n)\big)$$
$$= c\big(d(x_{n-1}, x_{n+1}) + d(x_n, x_n)\big)$$
$$\leq c\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big).$$

$$\therefore d(x_n, x_{n+1}) \leq \frac{c}{1-c} d(x_{n-1}, x_n).$$
$$\leq \delta d(x_{n-1}, x_n). \quad \lrcorner$$

Let $m, n \in \mathbb{N} : m \geq n$.
It holds that

$$
\begin{aligned}
& d(x_n, x_m) \\
& \leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \\
& \leq \delta^n d(x_0, x_1) + \cdots + \delta^{m-1} d(x_0, x_1) \\
& = \delta^n d(x_0, x_1) \frac{1}{1-\delta} \to 0 \ (m, n \to \infty).
\end{aligned}
$$

This indicates that
$\{x_n\}$ is a Cauchy sequence.
As X is complete, $\exists x^* \in X : x_n \to x^*$.

<u>$x^* = Tx^*$</u>

Proving this part
is a little tedious.
Wee omit it here.

<Uniqueness>
We omit it here.

//

Contraction-type Mappings

1. Give an example of a Kannan mapping.

2. Give an example of a contraction mapping that is not a Kannan mapping.

3. Prove the fixed point theorem for Kannan mappings.

4. Prove the fixed point theorem for Chatterjea mappings.

5. Although Kannan mappings and Chatterjea mappings are not continuous in general, they are continuous at fixed points. Prove these facts.

6. Demonstrate that contraction mappings, Kannan mappings, and Chatterjea mappings are special cases of Zamfirescu mappings.

7. Let X be a metric space and let $T : X \to X$ be a Zamfirescu mapping. For $x \in X$, define $x_n = T^n x$ for $n \in \mathbb{N} \cup \{0\}$. Then, $\{x_n\}$ is a Cauchy sequence. Prove this.

8. Let X be a metric space. Show that a Zamfirescu mapping $T : X \to X$ has at most only one fixed point.

9. Read Paper [4] and report its contents in the seminar.

10. Referring to the contents of the previous and two previous lectures, come up with and prove theorems about Kannan mappings and Chatterjea mappings by yourself.

11. Read Paper [3] and report its contents in the seminar.

References

[1] S.K. Chatterjea, "Fixed-point theorems," Dokladi na Bolgarskata Akademiya na Naukite 25(6) (1972).

[2] R. Kannan, "Some Results on Fixed Points," Bull. Calcutta Math. Soc. 10: (1968) 71-76.

[3] R. Kannan, "Some Results on Fixed Points-II," The American Mathematical Monthly, 76(4) (1969): 405-08.

[4] T. Zamfirescu, "Fix point theorems in metric spaces," Archiv der Mathmatik 23 (1972): 292-298.