Quasi-contraction mappings

warming up

X MS

$T: X \to X$

Def

$T: X \to X$ ρ-contraction

$\Leftrightarrow \exists \rho \in [0,1): \forall x, y \in X,$

$d(Tx, Ty) \leq \rho d(x,y)$

Def.

$M(x,n) \equiv \max\{d(x,Tx), d(x,T^2x), \dots, d(x,T^{n-1}x), d(x,T^n x)\}$

- $\forall x \in X.\ M(x,n) \leq M(x,n+1) \leq \cdots$
- $\forall x \in X, n \in \mathbb{N}.$

  $\exists k \in \{1, \dots, n\}: M(x,n) = d(x, T^k x)$

Lemma

X MS
$T : X \to X$ ρ-contraction
$x \in X$, $n \in \mathbb{N} \cup \{0\}$
$\Rightarrow M(x,n) \leq \frac{1}{1-\rho} d(x, Tx)$

Proof.

It follows that

$$M(x,n) \equiv \max\{d(x,Tx), \cdots, d(x,T^n x)\}$$
$$= d(x, T^k x) \text{ where } k \in \{1, \cdots, n\}$$
$$\leq d(x,Tx) + d(Tx, T^k x)$$
$$\leq d(x,Tx) + \rho\, d(x, T^{k-1}x)$$
$$\leq d(x,Tx) + \rho M(x,n).$$

We have $(1-\rho) M(x,n) \leq d(x,Tx)$.

As $1-\rho > 0$, $M(x,n) \leq \frac{1}{1-\rho} d(x,Tx)$.

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Th

X CMS

$T: X \to X$ ρ-contraction

$\Rightarrow \exists^1 x^* \in F(T): \forall x \in X,\ T^n x \to x^*$

Proof.

⟨Existence⟩

Let $x \in X$ and define $x_n = T^n x$ $(n \in \mathbb{N} \cup \{0\})$.

We show that $\{x_n\}$ is a Cauchy sequence.

Let $m, n \in \mathbb{N}: m > n$.

We have

$$
\begin{aligned}
& d(x_n, x_m) \\
&= d(x_n, T^{m-n} x_n) \\
&\leq M(x_n, m-n) \\
&= d(x_n, T^k x_n) \text{ where } k = 1, \cdots, m-n. \\
&= d(T x_{n-1}, T^{k+1} x_{n-1}) \\
&\leq \rho\, d(x_{n-1}, T^k x_{n-1}) \\
&\leq \rho\, M(x_{n-1}, m-n) \\
&\quad \cdots \\
&\leq \rho^n M(x, m-n) \\
&\leq \rho^n \frac{1}{1-\rho} d(x, Tx) \to 0 \quad (m, n \to \infty).
\end{aligned}
$$

⌟

As X is complete, $\exists x^* \in X : x_n \to x^*$.

<u>$x^* = Tx^*$</u>

As T is continuous,

$Tx^* = T(\lim x_n) = \lim Tx_n = \lim x_{n+1} = x^*$. $\lrcorner$

<Uniqueness>

Let $x^*, y^* \in F(T)$.

It holds that

$$d(x^*, y^*) = d(Tx^*, Ty^*)$$
$$\leqq \rho\, d(x^*, y^*).$$

$\therefore (1-\rho) d(x^*, y^*) \leqq 0$.

As $1-\rho > 0$, we obtain $x^* = y^*$. //

Def

X M.S

$T: X \to X$ <u>ρ-quasi-contraction</u>

$\Leftrightarrow \exists \rho \in [0,1) : \forall x, y \in X$

$$d(Tx, Ty) \le \rho \cdot \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(Tx,y)\}$$

$A \subset X, \neq \emptyset$

$\delta(A) = \sup\{d(x,y) \mid x, y \in A\}$

$T : X \to X$

$x \in X$

Def.

$$\boxed{\begin{aligned} &O(x,n) = \{x, Tx, T^2x, \cdots, T^nx\} \\ &\qquad\qquad\qquad (n \in \mathbb{N} \cup \{0\}) \\ &O(x,\infty) = \{x, Tx, T^2x, \cdots\} \end{aligned}}$$

- $\delta(O(x,0)) = 0$

  $\leq \delta(O(x,1)) \leq \delta(O(x,2)) \leq \cdots$

  $\forall x \in X$

- $\delta(O(x,\infty)) = \sup_{n \in \mathbb{N}} \delta(O(x,n))$

- $O(x,0)=\{x\}$

  $\therefore \delta(O(x,0))=0$

- $O(x,1)=\{x, Tx\}$

  $\therefore \delta(O(x,1))=d(x,Tx)$

- $O(x,2)=\{x, Tx, T^2x\}$

  $\therefore \delta(O(x,2))$

  $$=\max\{d(x,Tx), d(x,T^2x), d(Tx,T^2x)\}$$

- $O(x,3)=\{x, Tx, T^2x, T^3x\}$

  $\therefore \delta(O(x,3))$

  $$=\max\{d(x,Tx), d(x,T^2x), d(x,T^3x), d(Tx,T^2x), d(Tx,T^3x), d(T^2x,T^3x)\}$$

Lemma A

X MS

$T: X \to X$ ρ-quasi-contraction

$x \in X$, $n \in \mathbb{N}$

$\Rightarrow \forall i, j \in \{1, \cdots, n\}$,

$$d(T^i x, T^j x) \leq \rho \cdot \delta(O(x, \max\{i,j\}))$$
$$\leq \rho \cdot \delta(O(x,n))$$

Proof.

Note that

$T^{i-1}x, T^i x, T^{j-1}x, T^j x \in O(x, \max\{i,j\})$,

where $T^0 x = x$.

As T is ρ-quasi-contraction,

$$\begin{aligned}
&d(T^i x, T^j x) \\
&\leq \rho \cdot \max\{d(T^{i-1}x, T^{j-1}x), \\
&\qquad d(T^{i-1}x, T^i x), d(T^{j-1}x, T^j x), \\
&\qquad d(T^{i-1}x, T^j x), d(T^i x, T^{j-1}x)\} \\
&\leq \rho \cdot \delta(O(x, \max\{i,j\})) \\
&\leq \rho \cdot \delta(O(x,n)).
\end{aligned}$$

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Lemma B.

X MS
$T: X \to X$ ρ-quasi-contraction
$x \in X$, $n \in \mathbb{N}$

$$\Rightarrow \delta(O(x,n))$$
$$= \max\{d(x,Tx), d(x,T^2x), \cdots, d(x,T^nx)\} \quad (*)$$

Proof.

Suppose that $\delta(O(x,n)) = d(Tx, T^j x)$
for some $j \in \{1, \cdots, n\}$.

From Lemma A,

$$\delta(O(x,n)) = d(Tx, T^j x)$$
$$\leq \rho \cdot \delta(O(x,j))$$
$$\leq \rho \cdot \delta(O(x,n)).$$

$$\therefore (1-\rho)\delta(O(x,n)) \leq 0.$$

$$\therefore \delta(O(x,n)) = 0.$$

In this case, RHS of $(*) = 0$.

$$\therefore \text{LHS} = \text{RHS} \ (=0).$$

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Lemma C

X MS
$T : X \to X$ ρ-quasi-contraction
$x \in X$
$\Rightarrow \delta(O(x,\infty)) \leq \frac{1}{1-\rho} d(x, Tx)$

Proof.

As $\delta(O(x,\infty)) = \sup_{n \in \mathbb{N}} \delta(O(x,n))$,
it is sufficient to prove that
$\forall n \in \mathbb{N},\ \delta(O(x,n)) \leq \frac{1}{1-\rho} d(x, Tx)$.

From Lemma B,
$\exists k \in \{1, \cdots, n\} : d(x, T^k x) = \delta(O(x,n))$.

Using Lemma A, we have

$$\begin{aligned}
\delta(O(x,n)) &= d(x, T^k x) \\
&\leq d(x, Tx) + d(Tx, T^k x) \\
&\leq d(x, Tx) + \rho \cdot \delta(O(x,n))
\end{aligned}$$

Lemma A

$\therefore \delta(O(x,n)) \leq \frac{1}{1-\rho} d(x, Tx)$. //

Def.

X MS

$T : X \to X$

X : <u>T-orbitally complete</u>

$\Leftrightarrow$ $\{x_n\} \subset X$ Cauchy sequence,

$\exists x \in X : \{x_n\} \subset O(x, \infty)$

$= \{x, Tx, T^2x, \cdots\}$

$\Rightarrow \{x_n\}$: convergent

X complete

X T-orbitally complete

ex

$X = [0, 2)$

$T : X \to X$

$Tx = \frac{1}{2}x \quad {}^{\forall}x \in X$

Then, $O(x, \infty) = \left\{ x, \frac{1}{2}x, \frac{1}{4}x, \cdots, \frac{1}{2^{n-1}}x, \cdots \right\}$

where $x \in X$.

In this case, X is T-orbitally complete.

However, X is <u>not</u> complete.

Lemma A

X MS
$T: X \to X$ ρ-quasi-contraction
$x \in X$, $n \in \mathbb{N}$
$\Rightarrow \forall i, j \in \{1, \dots, n\}$,
$$d(T^i x, T^j x) \leq \rho \cdot \delta(O(x, \max\{i, j\})) \leq \rho \cdot \delta(O(x, n))$$

Lemma B.

X MS
$T: X \to X$ ρ-quasi-contraction
$x \in X$, $n \in \mathbb{N}$
$\Rightarrow \delta(O(x, n))$
$$\leq \max\{d(x, Tx), d(x, T^2 x), \dots, d(x, T^n x)\}$$

Lemma C.

X MS
$T: X \to X$ ρ-quasi-contraction
$x \in X$
$$\Rightarrow \delta(O(x, \infty)) \leq \frac{1}{1-\rho} d(x, Tx)$$

Th

$$\boxed{\begin{array}{l} X \text{ } T\text{-orbitally CMS} \\ T: X \to X \ \rho\text{-quasi-contraction} \\ \Rightarrow \exists^1 x^* \in F(T) : \forall x \in X,\ T^n x \to x^* \end{array}}$$

Proof.

<Existence>

Let $x \in X$ and define $x_n = T^n x$ $(n \in \mathbb{N} \cup \{0\})$.

$\{x_n\}$ is a Cauchy sequence.

Let $m, n \in \mathbb{N} : m \geq n$.

It follows that

$$\begin{aligned}
d(x_n, x_m) &= d(T^n x, T^m x) \\
&= d(T(T^{n-1}x), T^{m-n+1}(T^{n-1}x)) && \text{Lemma A} \\
&\leq \rho \cdot \delta(O(x_{n-1}, m-n+1)) && \text{Lemma B} \\
&= \rho \cdot d(x_{n-1}, T^k x_{n-1}) \\
&\quad \text{where } k \in \{1, 2, \cdots, m-n+1\} \\
&= \rho \cdot d(T x_{n-2}, T^{k+1} x_{n-2}) && \text{Lemma A} \\
&\leq \rho^2 \cdot \delta(O(x_{n-2}, k+1)) && 1 \leq k \leq m-n+1 \\
&\leq \rho^2 \cdot \delta(O(x_{n-2}, m-n+2)) \\
&\leq \cdots \leq \rho^n \cdot \delta(O(x, m)) \\
&\leq \rho^n \delta(O(x, \infty)) \underset{\text{Lemma C}}{\leq} \rho^n \frac{1}{1-\rho} d(x, Tx) \to 0.
\end{aligned}$$

As X is T-orbitally complete,

$\exists x^* \in X : x_n \to x^*$.

<u>$x^* = Tx^*$</u>

It holds that

$$d(x^*, Tx^*) \leq d(x^*, x_{n+1}) + d(Tx_n, Tx^*)$$

$$\leq d(x^*, x_{n+1}) + \rho \cdot \max\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), \underline{d(x_n, Tx^*)}, d(x_{n+1}, x^*)\}$$

$$\leq d(x^*, x_{n+1}) + \rho \cdot \max\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), \underline{d(x_n, x^*) + d(x^*, Tx^*)}, d(x_{n+1}, x^*)\}$$

$$\leq d(x^*, x_{n+1}) + \rho\,[d(x_n, x^*) + d(x_n, x_{n+1}) + d(x^*, Tx^*) + d(x_{n+1}, x^*)]$$

As $x_n \to x^*$, we obtain

$$d(x^*, Tx^*) \leq \rho\, d(x^*, Tx^*).$$

As $\rho \in [0,1)$, we have $d(x^*, Tx^*) \leq 0$.

$\therefore x^* = Tx^*$. $\lrcorner$

<Uniqueness>

Let $x^*, y^* \in F(T)$.

As T is ρ-quasi-contraction,

$$
\begin{aligned}
& d(x^*, y^*) \\
& = d(Tx^*, Ty^*) \\
& \leq \rho \cdot \max\{d(x^*, y^*), d(x^*, Tx^*), d(y^*, Ty^*), \\
& \qquad d(x^*, Ty^*), d(Tx^*, y^*)\} \\
& = \rho \cdot d(x^*, y^*).
\end{aligned}
$$

$\therefore (1-\rho) d(x^*, y^*) \leq 0.$

As $1-\rho > 0$, we obtain $x^* = y^*$. //

Quasi-contraction mappings

1. Let $T : X \to X$ be a ρ-contraction mapping, where is a Xmetric space. Let $x \in X$ and $n \in \mathbb{N} \cup \{0\}$. Then, it holds that $M(x,n) \leq \frac{1}{1-\rho} d(x, Tx)$, where

$$M(x,n) = \max\{d(x,Tx), d(x,T^2x), \cdots, d(x,T^nx)\}.$$

Prove this.

2. Prove the Banach contraction principle using the result of Problem 1.

3. For Kannan mappings or Chatterjea mappings, does the result of Problem 2 hold or not?

For $T : X \to X$, define

$$O(x,\ n) = \{x, Tx, T^2x, \cdots, T^nx\} \text{ and}$$
$$O(x,\ \infty) = \{x, Tx, T^2x, \cdots, T^nx, \cdots\}.$$

Furthermore, for a subset A of a metric space X, denote by $\delta(A)$ a diameter of A, i.e.,

$$\delta(A) = \sup\{d(x,y) : x,y \in A\}.$$

4. Let $T : X \to X$ be a ρ-quasi-contraction mapping, where X is a metric space. Let $x \in X$ and $n \in \mathbb{N} \cup \{0\}$. Then, it holds that

$$\begin{aligned} d(T^ix, T^jx) &\leq \rho\delta(O(x,\ \max\{i,j\})) \\ &\leq \rho\delta(O(x,\ n)) \end{aligned}$$

for all $i,j \in \{1,2,\cdots,n\}$. Prove this.

5. Let $T : X \to X$ be a ρ-quasi-contraction mapping, where X is a metric space. Let $x \in X$ and $n \in \mathbb{N} \cup \{0\}$. Then, it holds that

$$\delta(O(x,\ n)) = \max\{d(x,Tx), d(x,T^2x), \cdots, d(x,T^nx)\}.$$

Prove this.

6. Let $T : X \to X$ be a ρ-quasi-contraction mapping, where X is a metric space. For $x \in X$, the following inequality holds:

$$\delta(O(x,\ \infty)) \leq \frac{1}{1-\rho} d(x, Tx).$$

Prove this.

7. Let $T : X \to X$ be a ρ-quasi-contraction mapping, where X is a T-orbitally complete metric space. Prove that T has a unique fixed point $x^* \in F(T)$, and that $T^nx \to x^*$ (as $n \to \infty$) for any initial point $x \in X$.

Reference

[1] Lj B. Ćirić, "A generalization of Banach's contraction principle," Proceedings of the American Mathematical Society, 45(2) (1974): 267-273.