φ-contraction mappings

Def

$\varphi : [0,\infty) \to [0,\infty)$ comparison function

$\Leftrightarrow$ (1) φ: monotone increasing

(2) $\forall t > 0$, $\varphi^n(t) \to 0$ $(n \to \infty)$

ex

- $\varphi(t) = at$ $\forall t \geq 0$

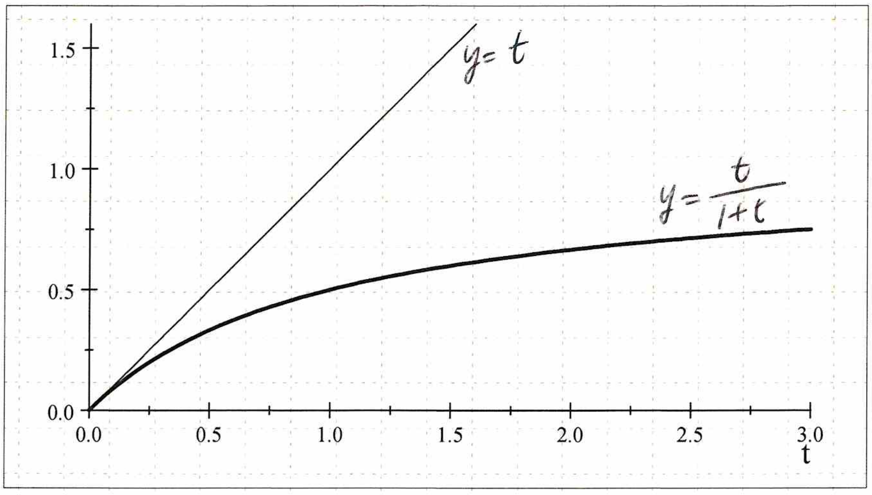
  where $a \in [0,1)$

- $\varphi(t) = \dfrac{t}{1+t}$ $\forall t \geq 0$

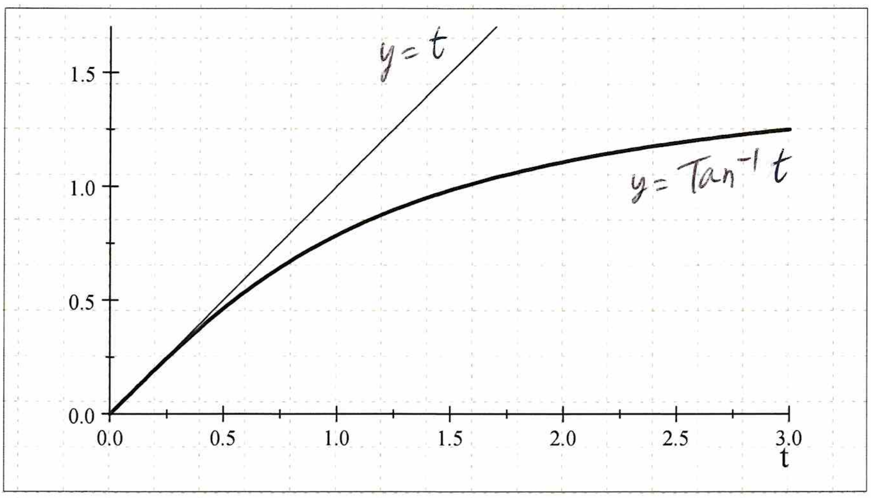
- $\varphi(t) = \mathrm{Tan}^{-1} t$ $\forall t \geq 0$

- $\varphi(t) = \log(1+t)$ $\forall t \geq 0$

$\varphi(t) = \frac{t}{1+t}$



$\varphi(t) = \arctan t$



$$\boxed{\begin{array}{l}\varphi : [0,\infty) \to [0,\infty) \text{ comparison function} \\ \Rightarrow \forall t>0,\ \varphi(t) < t\end{array}}$$

<u>Proof</u>

Let $t>0$.

Suppose by contradiction that

$(0<)\, t \leq \varphi(t)$.

From (1),

$$0 < t \leq \varphi(t) \leq \varphi^2(t).$$

Similarly, $0 < t \leq \varphi(t) \leq \varphi^n(t) \quad \forall n \in \mathbb{N}$.

From (2), $t = 0$.

This is a contradiction.

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Cor

$\varphi : [0,\infty) \to [0,\infty)$
comparison function

X : MS
$x, y \in X : x \neq y$
$\Rightarrow \varphi(d(x,y)) < d(x,y)$

Proof

As $x \neq y$, we have $d(x,y) > 0$.
Therefore, we obtain
$$\varphi(d(x,y)) < d(x,y).$$
//

$$\boxed{\begin{array}{l}\varphi : [0,\infty) \to [0,\infty) \\ \quad \text{comparison function} \\ \Rightarrow \varphi(0) = 0\end{array}}$$

Proof.

Suppose to lead a contradiction that

$$0 < \varphi(0).$$

As φ is monotone increasing,

$$\varphi(0) \leq \varphi^2(0).$$

Similarly, we have

$$0 < \varphi(0) \leq \varphi^n(0).$$

As $\varphi^n(t) \to 0$ ($\forall t > 0$) and $\varphi^n(0) = \varphi^{n-1}(\varphi(0))$,

we obtain $\varphi^n(0) \to 0$.

This contradicts $0 < \varphi(0)$.

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Cor

$\varphi : [0,\infty) \to [0,\infty)$ comparison function

$\Rightarrow \forall t \geqq 0, \ \varphi^n(t) \to 0$

Proof

If $t > 0$, then OK.

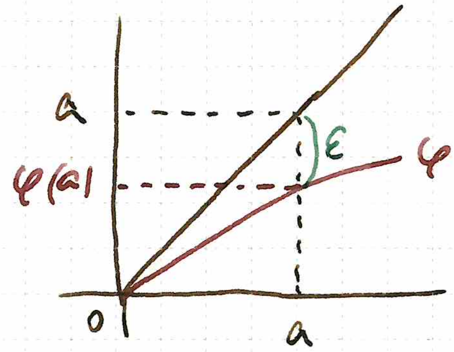
If $t = 0$, then

$$\{\varphi^n(t)\} = \{\varphi^n(0)\} = \{0, 0, 0, \cdots\}.$$

Thus, the desired result follows. //

$\varphi : [0,\infty) \to [0,\infty)$
continuous, monotone increasing
$\forall_{t>0},\ \varphi(t) < t$ — (*)
$\Longrightarrow \varphi^n(t) \to 0 \quad \forall_{t>0}$

Proof
Let $t>0$.
From (*), $\varphi(t) < t$.
As φ is monotone increasing,
$$\varphi^2(t) \leq \varphi(t) < t.$$
Similarly, we have
$$\cdots \leq \varphi^n(t) \leq \cdots \leq \varphi^2(t) \leq \varphi(t) < t. \quad \text{—(**)}$$
As $\{\varphi^n(t)\}$ is bdd below
with a lower bound $0 \in \mathbb{R}$,
we have that $\exists a \in \mathbb{R} : \varphi^n(t) \to a$.
We prove that $\underline{a=0}$.
Suppose to lead a contradiction that
$a>0$.
From (*), $\varphi(a) < a$.
Let $\varepsilon = a - \varphi(a) > 0$.



As φ is continuous, for $\varepsilon > 0$,

$\exists \delta > 0 : |x - a| < \delta \Rightarrow |\varphi(x) - \varphi(a)| < \varepsilon$.

i.e. $\varphi(a) - \varepsilon < \varphi(x) < \varphi(a) + \varepsilon = a$ — (*3)

As $\varphi^n(t) \to a$, for $\delta > 0$,

$\exists n_0 \in \mathbb{N} : |\varphi^{n_0}(t) - a| < \delta$.

From (*3), $\varphi^{n_0+1}(t) < a$

Using (**), we obtain

$\cdots \leq \varphi^{n_0+2}(t) \leq \varphi^{n_0+1}(t) < a$.

This contradicts $\varphi^n(t) \to a$.

$\therefore a = 0$.

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Cor

$\varphi : [0, \infty) \to [0, \infty)$

continuous, monotone increasing

$\varphi(t) < t \quad \forall t > 0$

$\Rightarrow \varphi$: comparison function

Def.

$T : X \to X$ <u>φ-contraction</u>

$\Leftrightarrow \exists \varphi : [0,\infty) \to [0,\infty)$ comparison function :

$\forall x, y \in X,\ d(Tx, Ty) \leq \varphi(d(x,y))$

ex

- $T : X \to X$ contraction
- $T : X \to X$ that satisfies
$$d(Tx, Ty) \leq \frac{1}{1 + d(x,y)} \cdot d(x,y) \quad \forall x, y \in X.$$
- $T : X \to X$
$$d(Tx, Ty) \leq \log(1 + d(x,y)).$$

<u>ex</u>

$X = [0,1]$

$T: X \to X$ defined by

$Tx = \log(1+x) \quad \forall x \in [0,1]$

$\Rightarrow T$ is φ-contraction

with $\varphi(x) = \log(1+x)$

<u>Proof</u>

We show that $\underline{d(Tx, Ty) \leq \varphi(d(x,y))}$.

i.e. $|\log(1+x) - \log(1+y)| \leq \log(1+|x-y|)$

Assume, w.l.g., that $x, y \in X : y \leq x$.

Then,

$$
\begin{aligned}
LHS &= \left| \log \frac{1+x}{1+y} \right| \\
&= \left| \log \frac{1+y-y+x}{1+y} \right| \\
&= \left| \log \left(1 + \frac{x-y}{1+y} \right) \right| \\
&\overset{x-y \geq 0}{=} \log \left(1 + \frac{x-y}{1+y} \right) \\
&\overset{y \geq 0}{\leq} \log(1 + (x-y)) \\
&= \log(1 + |x-y|). \\
&= RHS.
\end{aligned}
$$

//

$$\boxed{\begin{array}{l} T: X \to X \ \varphi\text{-contraction} \\ \Rightarrow T: \text{contractive} \end{array}}$$

Proof

Let $x, y \in X : x \neq y$.

We show that $d(Tx, Ty) < d(x, y)$.

As T is a φ-contraction,

$$d(Tx, Ty) \leq \varphi(d(x, y))$$

As $x \neq y$, $d(x, y) > 0$.

Thus, $d(Tx, Ty) \leq \varphi(d(x, y))$
$< d(x, y)$.

//

Cor

$$\boxed{\begin{array}{l} T: X \to X \ \varphi\text{-contraction} \\ \Rightarrow T: \text{cont}^{\text{i}}. \end{array}}$$

X MS

$T : X \to X$ φ-contraction

$x \in X$, $\varepsilon > 0$ $\to$ $\varphi(\varepsilon) < \varepsilon$

$d(x, Tx) < \varepsilon - \varphi(\varepsilon)$ — $(*)$

$\Longrightarrow B_\varepsilon(x)$: T-invariant

<u>Proof</u>

Let $y \in B_\varepsilon(x)$.

i.e. $d(x, y) < \varepsilon$ — $(**)$

We show that <u>$Ty \in B_\varepsilon(x)$.</u>

i.e. $d(x, Ty) < \varepsilon$.

As φ is monotone increasing,

we have from $(**)$ that $\varphi(d(x,y)) \leq \varphi(\varepsilon)$. —$(*3)$

The following holds:

$$
\begin{aligned}
d(x, Ty) &\leq d(x, Tx) + d(Tx, Ty) \\
&\leq d(x, Tx) + \varphi(d(x,y)) \\
&< \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon) \qquad (*)\ (*3) \\
&= \varepsilon .
\end{aligned}
$$

//

Th

X CMS

$T: X \to X$ φ-contraction

$\Rightarrow \exists^1 x^* \in F(T): {}^\forall x \in X, T^n x \to x^*$

Proof

<Existence>

Let $x \in X$, and define $x_n = T^n x$ $({}^\forall n \in \mathbb{N} \cup \{0\})$.

$\{x_n\} (\subset X)$: Cauchy seq.

It holds that

$$
\begin{aligned}
d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
&\leqq \varphi(d(x_{n-1}, x_n)) \\
&\leqq \cdots \\
&\leqq \varphi^n(d(x_0, x_1)) \longrightarrow 0 \quad (n \to \infty). \quad -①
\end{aligned}
$$

Let $\varepsilon > 0$.

Define $\delta(\varepsilon) = \varepsilon - \varphi(\varepsilon)$.

As $\varepsilon > 0$, we have $\varphi(\varepsilon) < \varepsilon$.

$\therefore \delta(\varepsilon) = \varepsilon - \varphi(\varepsilon) > 0$. $\quad$ —②

As $\varphi(\varepsilon) \in [0, \infty)$, $\delta(\varepsilon) = \varepsilon - \varphi(\varepsilon) \leqq \varepsilon$. $\quad$ —③

From ①, for ② $\delta(\varepsilon) > 0$,

$\exists n_0 \in \mathbb{N} : n \geqq n_0 \Rightarrow d(x_n, x_{n+1}) < \delta(\varepsilon)$.

$\therefore d(T^n x, T(T^n x))$

$= d(x_n, x_{n+1}) < \delta(\varepsilon) = \varepsilon - \varphi(\varepsilon) \overset{③}{\leqq} \varepsilon \quad (n \geqq n_0)$. —④

Thus, $B_\varepsilon(T^{n_0}x)$ is T-invariant.

From ④, $x_{n_0+1} \in B_\varepsilon(x_{n_0})$.

$$x_{n_0+1} = Tx_{n_0}$$

Consequently, $T^2 x_{n_0} = x_{n_0+2} \in B_\varepsilon(x_{n_0})$.

Similarly, $x_m, x_n \in B_\varepsilon(x_{n_0}) \quad (m, n \geqq n_0)$.

$\therefore d(x_m, x_n) < 2\varepsilon$.

$\therefore \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} : m, n \geqq n_0 \Rightarrow d(x_m, x_n) < 2\varepsilon$.

This shows that $\{x_n\}$ is a Cauchy seq. ⌟

As X is complete, $\exists x^* \in X : x_n \to x^*$.

As T is conti., $x^* = Tx^*$. ⌟

$\langle$Uniqueness$\rangle$

Let $x^*, y^* \in F(T) : x^* \neq y^*$.

Then,

$$
\begin{aligned}
& d(x^*, y^*) \\
&= d(Tx^*, Ty^*) \\
&\leq \varphi(d(x^*, y^*)) \\
&< d(x^*, y^*), \qquad (3)
\end{aligned}
$$

which is a contradiction.

//

φ –Contractions

1. Show the following:
Let $\varphi : [0,\infty) \to [0,\infty)$ be a comparison function. Then, for any $t > 0$, it holds true that $\varphi(t) < t$.

2. Show the following:
Let $\varphi : [0,\infty) \to [0,\infty)$ be a comparison function. Then, $\varphi(0) = 0$.

3. Show the following:
Let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous and monotone increasing function. Suppose that $\varphi(t) < t$ for any $t > 0$. Then, $\varphi : [0,\infty) \to [0,\infty)$ is a comparison function.

4. Give examples of comparison functions $\varphi : [0,\infty) \to [0,\infty)$.

5. Give examples of φ–contraction mappings defined on a metric space X.

6. Show the following:
Let X be a metric space and let $T : X \to X$ be a φ–contraction mapping. Then, T is contractive.

7. Prove the following lemma:
Lemma.
Let X be a metric space and let $T : X \to X$ be a φ–contraction mapping. Let $x \in X$ and $\varepsilon > 0$ such that $d(x, Tx) < \varepsilon - \varphi(\varepsilon)$. Then, an open ball (sphere) $B_\varepsilon(x)$ is T–invariant.

8. State the claim of the fixed point theorem for φ–contraction mappings and prove it.