Fixed points of a new type of contractive mappings in complete metric spaces

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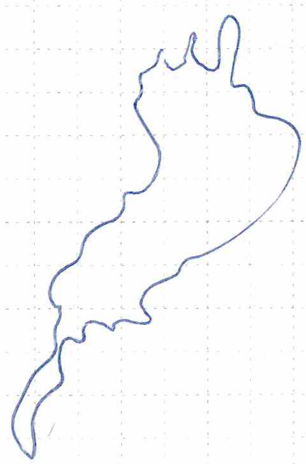
Fixed Point Theory and Applications
(2012)



$$\{a_n\} \subset [0,\infty)$$
$$\sum_{n=1}^{\infty} a_n < \infty$$
$$\Longrightarrow \sum_{k=n}^{\infty} a_k \to 0 \quad (n \to \infty)$$

<u>Proof</u>

It holds that

$$\sum_{k=n}^{\infty} a_k = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n-1} a_k \quad \forall n \in \mathbb{N}.$$

As RHS $\to 0$ $(n \to \infty)$,

we obtain the desired result.

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$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n} = \infty}$$

Proof

It holds that

$$n \leqq x \iff \frac{1}{n} \geqq \frac{1}{x}. \quad {}^{\forall} n \in \mathbb{N}.$$

Thus,
$$\frac{1}{n} = \int_n^{n+1} \frac{1}{n}\, dx$$
$$\geqq \int_n^{n+1} \frac{1}{x}\, dx$$

Summing these inequalities w.r.t. $n = 1, \cdots, N-1$,

we have
$$\sum_{n=1}^{N-1} \frac{1}{n} \geqq \int_1^N \frac{1}{x}\, dx$$
$$= [\log x]_1^N = \log N. \quad {}^{\forall} N \in \mathbb{N}.$$

As $N \to \infty$, we obtain $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

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Alternative Proof

It suffices to prove that

$$\sum_{n=1}^{2^k} \frac{1}{n} > 1 + \frac{k}{2} \quad \forall k \in \mathbb{N} \cup \{0\}.$$

(i) If $k=0$, the inequality holds.

(ii) Assume that $\displaystyle\sum_{n=1}^{2^k} \frac{1}{n} > 1 + \frac{k}{2}$.

Then,

$$\sum_{n=1}^{2^{k+1}} \frac{1}{n} = \sum_{n=1}^{2\cdot 2^k} \frac{1}{n}$$

$$= \sum_{n=1}^{2^k} \frac{1}{n} + \sum_{n=2^k+1}^{2\cdot 2^k} \frac{1}{n}$$

$$> 1 + \frac{k}{2} + \underbrace{\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2\cdot 2^k}}_{2^k \text{個}}$$

$$> 1 + \frac{k}{2} + \frac{2^k}{2\cdot 2^k}$$

$$= 1 + \frac{k+1}{2}.$$

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cf.

$$\boxed{\begin{array}{l}\{x_n\} \subset X \text{ Cauchy seq.} \\ \text{i.e. } d(x_n, x_m) \to 0 \quad (m, n \to \infty) \\ \Longrightarrow d(x_n, x_{n+1}) \to 0\end{array}}$$

$\not\Longleftarrow$

ex

Let $\begin{cases} X = \mathbb{R} \\ x_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \end{cases}$

Then, $x_{n+1} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1}$

Thus, $d(x_n, x_{n+1}) = |x_n - x_{n+1}| = \frac{1}{n+1} \to 0 \ (n \to \infty)$.

However, for $m > n$,

$$d(x_n, x_m) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m} \to \infty$$

as $m \to \infty$

with fixing $n \in \mathbb{N}$.

$\therefore d(x_n, x_m) \not\to 0$ as $m, n \to \infty$.

$f : [1, \infty) \to [0, \infty)$ conti., decreasing

$\Rightarrow$ Equivalent

① $\displaystyle\sum_{n=1}^{\infty} f(n) < \infty$

② $\displaystyle\int_1^{\infty} f(x)\,dx < \infty$

Proof

As f is decreasing,

$$n \leqq x \leqq n+1$$

$$\Rightarrow f(n) \geqq f(x) \geqq f(n+1).$$

Thus,
$$\begin{aligned} f(n) &= \int_n^{n+1} f(n)\,dx \\ &\geqq \int_n^{n+1} f(x)\,dx \\ &\geqq \int_n^{n+1} f(n+1)\,dx \\ &= f(n+1) \qquad \forall n \in \mathbb{N}. \end{aligned}$$

It follows that

$$\sum_{n=1}^{N-1} f(n) \geqq \int_1^N f(x)\,dx \geqq \sum_{n=2}^{N} f(n) \qquad \forall N \in \mathbb{N}.$$

From this, we obtain the desired result.

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$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^s} \begin{cases} = \infty & \text{if } s \leq 1 \\ \in \mathbb{R} & \text{if } s > 1 \end{cases}}$$

<u>Proof</u>

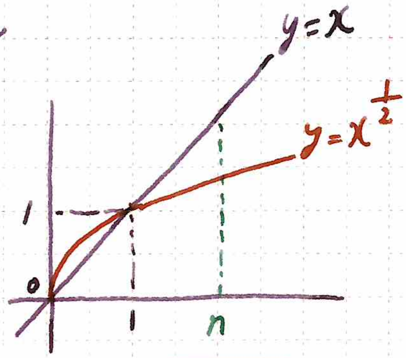
(i) If $s=1$, then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

If $s<1$, then $n \geqq n^s$, and hence,

$$\frac{1}{n} \leqq \frac{1}{n^s}.$$

Thus, $$\infty = \sum_{n=1}^{\infty} \frac{1}{n} \leqq \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This means that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \infty$. $\lrcorner$



$$\boxed{n^{\frac{1}{2}} \leqq n}$$

(ii) Assume that $s>1$.

We prove that $\displaystyle\underline{\int_1^{\infty} \frac{1}{x^s}\,dx \in \mathbb{R}}$.

It holds that

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^s}\,dx &= \lim_{k\to\infty} \int_1^{k} x^{-s}\,dx \\ &= \lim_{k\to\infty} \left[\frac{1}{1-s} x^{1-s}\right]_1^k \\ &= \lim_{k\to\infty} \left(\frac{1}{1-s}\frac{1}{k^{\boxed{s-1}_{(+)}}} - \frac{1}{1-s}\right) \\ &= \frac{1}{s-1} \in \mathbb{R}. \end{aligned}$$

Consequently, we have $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} \in \mathbb{R}$ if $s>1$.

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Def.

$F:(0,\infty)\to\mathbb{R}$ F-mapping

$\Leftrightarrow$ (F1) F is strictly increasing.

(F2) $r\to 0 \Leftrightarrow F(r)\to -\infty$

(F3) $\exists k\in(0,1): r^k F(r)\to 0 \ (r\to 0)$

ex

$F_1(x)=\log x$

$F_2(x)=\log x + x$

$F_3(x)=\log x + a \quad (a\in\mathbb{R}: \text{constant})$

$F_4(x)=-\dfrac{1}{\sqrt{x}}$

Def.

X MS

$T: X \to X$ <u>F-contraction</u>

$\Leftrightarrow \exists F: (0,\infty) \to \mathbb{R}$ F-mapping

$\exists \tau > 0 : \forall x, y \in X : Tx \neq Ty,$

$\tau + F(d(Tx, Ty)) \leq F(d(x, y))$

※ $\boxed{Tx \neq Ty} \Longrightarrow d(Tx, Ty) > 0$

$\Rightarrow F(d(Tx, Ty)) \in \mathbb{R}$ exists

$\Downarrow$

$x \neq y \Rightarrow d(x, y) > 0$

$\Rightarrow F(d(x, y)) \in \mathbb{R}$ exists

X MS

$T: X \to X$

$\Rightarrow$ Equivalent

① T: contraction

i.e. $\exists r \in (0,1) : \forall x, y \in X, d(Tx, Ty) \leq r\, d(x,y)$

② T: log-contraction

i.e. $\exists \tau > 0 : \forall x, y \in X : Tx \neq Ty,$

$\tau + \log d(Tx, Ty) \leq \log d(x,y)$

Proof

①$\Rightarrow$②

Define $\tau = -\log r > 0$.

Let $x, y \in X : Tx \neq Ty$.

As $Tx \neq Ty$, $d(Tx, Ty) > 0$.

Furthermore, we have $x \neq y$. $\therefore d(x,y) > 0$.

Therefore, $\log d(Tx, Ty), \log d(x,y) \in \mathbb{R}$ exists.

From ①, $d(Tx, Ty) \leq r\, d(x,y)$.

Hence, $\log d(Tx, Ty) \leq \log r + \log d(x,y)$.

$\therefore -\log r + \log d(Tx, Ty) \leq \log d(x,y)$.

$\therefore \tau + \log d(Tx, Ty) \leq \log d(x,y)$. $\lrcorner$

② ⇒ ①

Define $r \equiv e^{-\tau}$.

As $\tau > 0$, it follows that $r \equiv e^{-\tau} \in (0,1)$.

Let $x, y \in X$.

Our goal is to prove that

$$d(Tx, Ty) \leq r\, d(x, y).$$

If $Tx = Ty$, then the desired result follows.

Assume that $Tx \neq Ty$. Then, $x \neq y$.

From ②, $\tau + \log d(Tx, Ty) \leq \log d(x, y)$.

Consequently,

$$\log \frac{d(Tx, Ty)}{d(x, y)} \leq -\tau$$

$$\therefore \frac{d(Tx, Ty)}{d(x, y)} \leq e^{-\tau} \equiv r$$

Thus, we have $d(Tx, Ty) \leq r\, d(x, y)$.

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$T : X \to X$ F-contraction

$\Rightarrow \forall x, y \in X : x \neq y,$

$d(Tx, Ty) < d(x, y)$

Proof

Let $x, y \in X : x \neq y$.

We show that $\underline{d(Tx, Ty) < d(x, y)}$.

(i) $\underline{Tx = Ty}$ OK

(ii) $\underline{Tx \neq Ty}$

As T is F-contraction,

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y))$$

for some $\tau > 0$.

As $\tau > 0$,

$$F(d(Tx, Ty)) < F(d(x, y)).$$

As F is monotone increasing, we have

$$d(Tx, Ty) < d(x, y).$$

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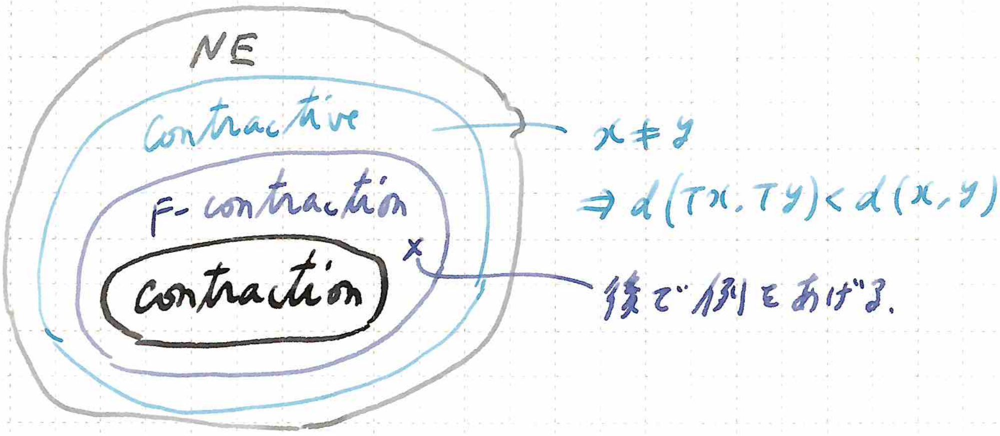

Cor

X metric space

$T : X \to X$ F-contraction

$\Rightarrow T$: conti.

Th

X CMS
$T: X \to X$ F-contraction
$\Rightarrow \exists^1 x^* \in F(T) : \forall x \in X,\ T^n x \to x^*$

Proof

<Existence>

Let $x = x_0 \in X$.

Define $\begin{cases} x_n = T^n x & \forall n \in \mathbb{N} \\ r_n = d(x_n, x_{n+1}) & \forall n \in \mathbb{N} \cup \{0\}. \end{cases}$

Our aim is to prove that $\{x_n\}$ is a Cauchy seq.

If $r_n \equiv d(x_n, x_{n+1}) = 0$ for some $n \in \mathbb{N} \cup \{0\}$,

then
$$\begin{aligned} x_{n+2} &= T x_{n+1} \\ &= T x_n = x_{n+1}. \end{aligned}$$

Thus, $x_n = x_{n+1} = x_{n+2} = \cdots$.

Therefore, we assume, w.l.g., that $r_n > 0\ \forall n \in \mathbb{N} \cup \{0\}$.

We show that $\underline{r_n \to 0.}$

It follows that
$$\begin{aligned} F(r_n) &= F(d(x_n, x_{n+1})) \\ &= F(d(Tx_{n-1}, Tx_n)) \\ &\leq F(d(x_{n-1}, x_n)) - \tau \\ &= F(r_{n-1}) - \tau \\ &\leq \cdots \leq F(r_0) - n\tau. \qquad \text{— ①} \end{aligned}$$

$F : (0, \infty) \to \mathbb{R}$
(F1) F: monotone increasing
(F2) $\alpha_n \to 0 \Leftrightarrow F(\alpha_n) \to -\infty$
(F3) $\exists k \in (0,1)$:
$\lim_{\alpha \downarrow 0} \alpha^k F(\alpha) = 0$

From ①, $F(\gamma_n) \to -\infty$.

From (F2), $\gamma_n \to 0$.

From (F3), $\gamma_n^k F(\gamma_n) \to 0$ for some $k \in (0,1)$. — ②

Using ①, we have

$$F(\gamma_n) - F(\gamma_0) \leq -n\tau$$

$$\therefore \gamma_n^k \left(F(\gamma_n) - F(\gamma_0) \right) \leq \gamma_n^k (-n\tau) \leq 0.$$

From ②, $n\gamma_n^k \to 0$. as $n \to \infty$.

For $1 > 0$, $\exists n_0 \in \mathbb{N}$:

$$n \geq n_0 \Rightarrow \gamma_n < \frac{1}{n^{1/k}} \quad — ③$$

$\{x_n\}$: Cauchy seq.

Let $m, n \in \mathbb{N}$: $n_0 \leq n < m$.

From ③, $d(x_n, x_m) \leq \gamma_n + \gamma_{n+1} + \cdots + \gamma_{m-1}$

$$< \sum_{l=n}^{\infty} \gamma_l$$

$$< \sum_{l=n}^{\infty} \frac{1}{l^{1/k}} \quad ③$$

As $k \in (0,1)$, $\displaystyle\sum_{l=n}^{\infty} \frac{1}{l^{1/k}} < \infty$ for $n \geq n_0$.

Therefore, $d(x_n, x_m) \to 0$ $(n, m \to \infty)$.

As X is complete,

$\exists x^* \in X : x_n \to x^*$

$x^* \in F(T)$.

It follows that

$$
\begin{aligned}
Tx^* &= T(\lim x_n) \\
&= \lim Tx_n \\
&= \lim x_{n+1} \\
&= x^*.
\end{aligned}
$$

<Uniqueness>

Let $x^*, y^* \in F(T) : x^* \neq y^*$.

Then, $Tx^* = x^* \neq y^* = Ty^*$.

As T is F-contraction,

$$
\begin{aligned}
\tau &\leq F(d(x^*, y^*)) - F(d(\underline{Tx^*}, \underline{Ty^*})) \\
&= F(d(x^*, y^*)) - F(d(\underline{x^*}, \underline{y^*})) \\
&= 0.
\end{aligned}
$$

This contradicts $\tau > 0$.

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$r_n = d(x_{n+1}, x_n)$

$r_n \to 0$ だけからは、$\{x_n\}$: Cauchy seq.

は言えない。

<u>ex</u>

$$a_n = \sum_{k=1}^{n} \frac{1}{k}$$

Then, $|a_n - a_{n+1}| = \frac{1}{n+1} \to 0$.

However,

$$|a_m - a_n| = \sum_{k=n+1}^{m} \frac{1}{k} \quad \text{where } m > n.$$

$\to \infty$ as $m \to \infty$

with $n \in \mathbb{N}$ is fixed.

$F_1, F_2 : (0,\infty) \to \mathbb{R}$ F-mappings
$G \equiv F_2 - F_1 : (0,\infty) \to \mathbb{R}$ nondecreasing
$T : X \to X$ F_1-contraction
$\Rightarrow T$: F_2-contraction

Proof.

Let $x, y \in X : Tx \neq Ty$.

Then, $x \neq y$, and

$F_i(d(Tx, Ty))$, $F_i(d(x,y))$ $(i=1,2)$,
$G(d(Tx,Ty))$, $G(d(x,y))$ are defined.

Our aim is to prove that

$\exists \tau > 0$ that is independent from x, y :

$$\tau + F_2(d(Tx,Ty)) \leq F_2(d(x,y)).$$

As T is a F_1-contraction,

$\exists \tau > 0$ that is independent from x, y :

$$\tau + F_1(d(Tx,Ty)) \leq F_1(d(x,y)). \quad \text{— ①}$$

As T is a F_1-contraction, it is contractive.

As $x \neq y$, we have

$$d(Tx, Ty) < d(x,y).$$

As G is nondecreasing,

$$G(d(Tx,Ty)) \leq G(d(x,y)). \quad \text{— ②}$$

It holds that

$$\tau + F_2(d(Tx, Ty))$$

$$= \tau + F_1(d(Tx, Ty)) + G(d(Tx, Ty))$$

$$\leqq F_1(d(x, y)) + G(d(x, y))$$

$$= F_2(d(x, y)).$$

$G \equiv F_2 - F_1$

$\therefore F_2 = F_1 + G$

① ②

$F_2 = F_1 + G$

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$$\boxed{\begin{array}{l} T: X \to X \text{ contraction} \\ F_2 : (0,\infty) \to \mathbb{R} \\ \quad F_2(x) = \log x + x \quad \forall x \in (0,\infty) \\ \Rightarrow T : F_2\text{-contraction} \end{array}}$$

Proof

As T is a contraction, it is log-contraction.

Define $G(x) = F_2(x) - \log x = x$.

Then, G is nondecreasing

Therefore, T is F_2-contraction.

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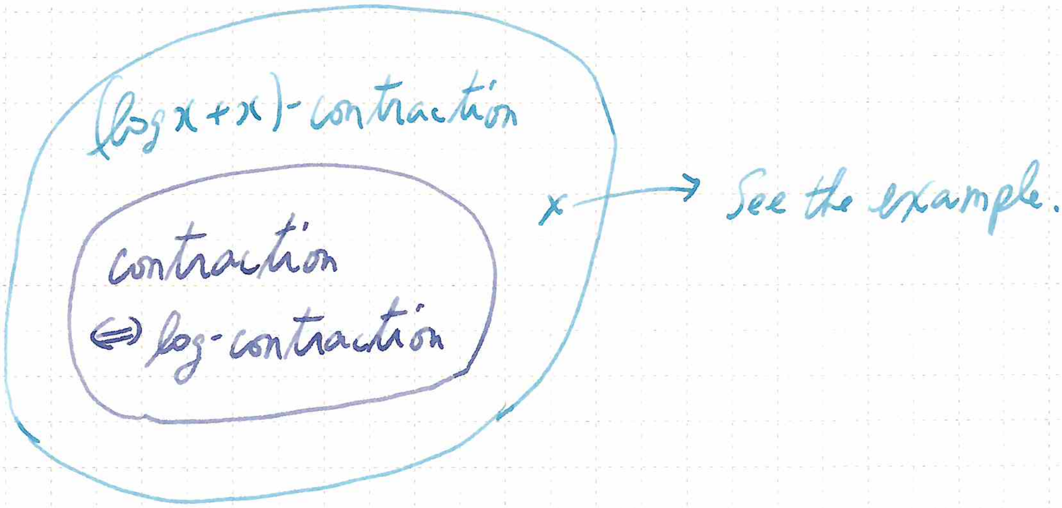


- $F(x) = \log x$

Let $x, y \in X : Tx \neq Ty$. $\rightarrow x \neq y$

Then,

$$\tau + F(d(Tx, Ty)) \leq F(d(x,y))$$

$$\Leftrightarrow \tau + \log d(Tx, Ty) \leq \log d(x,y)$$

$$\Leftrightarrow \log \frac{d(Tx, Ty)}{d(x,y)} \leq -\tau$$

$$\Leftrightarrow \underline{d(Tx, Ty) \leq e^{-\tau} d(x,y).}$$

Note that $e^{-\tau} \in (0,1)$.

- $F(x) = \log x + x$

Let $x, y \in X : Tx \neq Ty$

Then,

$$\tau + F(d(Tx, Ty)) \leq F(d(x,y))$$

$$\Leftrightarrow \tau + \log d(Tx, Ty) + d(Tx, Ty) \leq \log d(x,y) + d(x,y)$$

$$\Leftrightarrow \log \frac{d(Tx, Ty)}{d(x,y)} + d(Tx, Ty) - d(x,y) \leq -\tau$$

$$\Leftrightarrow \underline{\frac{d(Tx, Ty)}{d(x,y)} e^{d(Tx, Ty) - d(x,y)} \leq e^{-\tau}}$$

<u>ex</u>

$S_1 = 1$

$S_2 = 1+2$

$\cdots$

$S_n = 1+2+\cdots+n = \dfrac{n(n+1)}{2}$

$\cdots$

Let $X = \{S_n \mid n \in \mathbb{N}\}$ $(\subset \mathbb{N} \subset \mathbb{R})$.

Then, (X, d) is a complete metric space,

where $d(x, y) = |x-y|$.

$T : X \to X$ defined as follows:

$$T(S_n) = \begin{cases} S_{n-1} & \text{for } n = 2, 3, \cdots \\ S_1 & \text{for } n = 1. \end{cases}$$

Clearly, $F(T) = \{S_1\}$.

Let $\begin{cases} F_1(x) = \log x \\ F_2(x) = \log x + x \end{cases}$.

- T is <u>not</u> a F_1-contraction.

i.e. $\nexists \tau > 0$: $\forall x, y \in X$: $Tx \neq Ty$,

$$\frac{d(Tx, Ty)}{d(x,y)} \leq e^{-\tau}.$$

Let $n \in \mathbb{N} : n \geq 3$.

Then, $T(s_n) = s_{n-1} \neq s_1 = T(s_2) = T(s_1)$

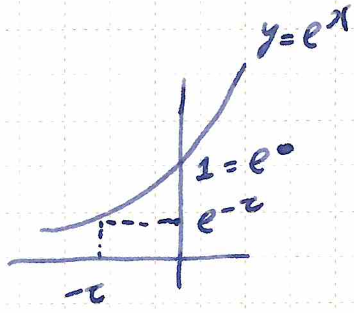
It holds that

$$\frac{d(Ts_n, Ts_1)}{d(s_n, s_1)}$$

$$= \frac{d(s_{n-1}, s_1)}{d(s_n, s_1)}$$

$$= \frac{\frac{n(n-1)}{2} - 1}{\frac{n(n+1)}{2} - 1}$$

$$= \frac{n^2 - n - 2}{n^2 + n - 2} \longrightarrow 1 = e^0$$



$\therefore \forall \tau > 0, \exists n \in \mathbb{N}$: sufficiently large :

$T(s_n) \neq T(s_1)$,

$$\frac{d(Ts_n, Ts_1)}{d(s_n, s_1)} > e^{-\tau}.$$

• T is F_2-contraction with $\tau = 1$.

i.e. $$\frac{d(Tx,Ty)}{d(x,y)} e^{d(Tx,Ty)-d(x,y)} \leqq e^{-1}$$

for $x, y \in X$: $Tx \neq Ty$.

Note that $TS_m \neq TS_n$

$$\Longleftrightarrow \begin{cases} \text{(i) } m > n > 1 \text{ or} \\ \text{(ii) } m \geqq 3,\ n = 1. \end{cases}$$

(i) $\underline{m > n > 1}$

It holds that

$$\frac{d(TS_m, TS_n)}{d(S_m, S_n)} e^{d(TS_m, TS_n) - d(S_m, S_n)}$$

$$= \frac{d(S_{m-1}, S_{n-1})}{d(S_m, S_n)} e^{d(S_{m-1}, S_{n-1}) - d(S_m, S_n)}$$

$$= \frac{\frac{m(m-1)}{2} - \frac{n(n-1)}{2}}{\frac{m(m+1)}{2} - \frac{n(n+1)}{2}} e^{(S_{m-1} - S_{n-1}) - (S_m - S_n)}$$

$$= \frac{m^2 - n^2 - (m-n)}{m^2 - n^2 + (m-n)} e^{n-m}$$

$$= \frac{m+n-1}{m+n+1} e^{n-m}$$

$$\leqq e^{n-m} \leqq e^{-1}.$$

$\lrcorner$

(ii) $m \geqq 3,\ n=1$

Then,

$$\frac{d(T s_m, T s_1)}{d(s_m, s_1)} e^{d(T s_m, T s_1) - d(s_m, s_1)}$$

$$= \frac{d(s_{m-1}, s_1)}{d(s_m, s_1)} e^{d(s_{m-1}, s_1) - d(s_m, s_1)}$$

$$= \frac{\frac{m(m-1)}{2} - 1}{\frac{m(m+1)}{2} - 1} e^{(s_{m-1} - 1) - (s_m - 1)}$$

$$= \frac{m^2 - m - 2}{m^2 + m - 2} e^{-m}$$

$$\leqq e^{-m} \leqq e^{-3} < e^{-1}.$$

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F−Contractions

1. Show that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

2. Show the following:
Let $f : [1,\infty) \to [0,\infty)$ be a continuous and decreasing function. Then, the following two statement is equivalent:
(1) $\sum_{n=1}^{\infty} f(n) < \infty$;
(2) $\int_{1}^{\infty} f(x)dx < \infty$.

3. Show the following:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \begin{cases} = \infty & \text{if } s \leq 1; \\ < \infty & \text{if } s > 1. \end{cases}$$

4. Show that the following functions are F−functions:
(1) $F(x) = \log x + x$, (2) $F(x) = -1/\sqrt{x}$.

5. Let T be a self−mapping defined on a metric space X. Show that T is a contraction if and only if T is a log−contraction.

6. Let T be a self−mapping defined on a metric space X. Show that a F−contraction mapping is contractive.

7. State the claim of the fixed point theorem for F−contraction mappings, and prove it.

8. Read Paper [2] and report its contents in the seminar.

References

[1] D. Wardowski, "Fixed points of a new type of contractive mappings in complete metric spaces," Fixed point theory and applications 2012(1) (2012): 1−6.
[2] D. Wardowski and N. Van Dung, "Fixed points of F−weak contractions on complete metric spaces," Demonstratio Mathematica, 47(1) (2014), 146−155.