Alternative proofs for the Banach contraction principle

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$T: X \to X$ r-contraction $(0<r<1)$

$x \in X$, $\varepsilon > 0$

$d(x, Tx) \leqq (1-r)\varepsilon$ — (*)

$\Rightarrow B_\varepsilon(x)$: T-invariant

Proof

Let $y \in B_\varepsilon(x)$

i.e. $d(x,y) < \varepsilon$ — (**)

We show that $\underline{Ty \in B_\varepsilon(x)}$.

i.e. $d(x, Ty) < \varepsilon$.

It holds true that

$$\begin{aligned} d(x, Ty) &\leqq d(x, Tx) + d(Tx, Ty) \\ &\leqq (1-r)\varepsilon + r\,d(x,y) && (*) \\ &< (1-r)\varepsilon + r\varepsilon && (**) \\ &= \varepsilon. \end{aligned}$$

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X CMS
$T: X \to X$ r-contraction $(0<r<1)$
$\Rightarrow \exists^1 x^* \in F(T) : {}^\forall x \in X,\ T^n x \to x^*$

Proof
Let $x \in X$, and define $x_n = T^n x$ $(n \in \mathbb{N} \cup \{0\})$.

$\{x_n\} (\subset X)$: Cauchy seq.

It holds that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq r\, d(x_{n-1}, x_n) \leq \cdots \leq r^n d(x_0, x_1) \to 0. \quad \text{— } (*)$$

Let $\varepsilon > 0$, and define $\delta = (1-r)\varepsilon > 0$.

From $(*)$, for $\delta \left(= (1-r)\varepsilon\right) > 0$,

$${}^\exists n_0 \in \mathbb{N} : d(x_{n_0}, Tx_{n_0}) < \delta = (1-r)\varepsilon \leq \varepsilon. \quad (**)$$

Thus, $B_\varepsilon(x_{n_0})$ is T-invariant.

From $(**)$, $Tx_{n_0} = x_{n_0+1} \in B_\varepsilon(x_{n_0})$.

$\therefore Tx_{n_0+1} = x_{n_0+2} \in B_\varepsilon(x_{n_0})$.

$\therefore \{x_{n_0}, x_{n_0+1}, x_{n_0+2}, \cdots\} \subset B_\varepsilon(x_{n_0})$.

Therefore, ${}^\forall m, n \geq n_0$, $d(x_m, x_n) < 2\varepsilon$.

This indicates that $\{x_n\}$ is a Cauchy seq. ⌟

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X CMS
$T : X \to X$ r-contraction $(0<r<1)$
$\Rightarrow \exists^1 x^* \in F(T) : \forall x \in X,\ T^n x \to x^*$

Proof

Let $x \in X$, and define $x_n = T^n x$ $(n \in \mathbb{N} \cup \{0\})$.
Assume, w.l.g., that $x_n \neq x_{n+1}$ $\forall n \in \mathbb{N} \cup \{0\}$.
It holds that

$$
\begin{aligned}
d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
&\leqq r\, d(x_{n-1}, x_n).
\end{aligned}
$$

Then,

$$
\begin{aligned}
\log d(x_n, x_{n+1}) &\leqq \log d(x_{n-1}, x_n) + \log r \\
&\leqq \log d(x_{n-2}, x_{n-1}) + 2 \log r \\
&\leqq \cdots \\
&\leqq \log d(x_0, x_1) + n \underbrace{\log r}_{<0} \longrightarrow -\infty \qquad \text{— } (*)
\end{aligned}
$$

$\therefore \log d(x_n, x_{n+1}) \to -\infty$

$\therefore d(x_n, x_{n+1}) \to 0.$ — $(**)$

From $(*)$,

$$\sqrt{d(x_n, x_{n+1})}\left(\log d(x_n, x_{n+1}) - \log d(x_0, x_1)\right)$$

$$\leqq \underbrace{n \log r}_{<0} \cdot \sqrt{d(x_n, x_{n+1})} \leqq 0$$

From $(**)$, $n\sqrt{d(x_n, x_{n+1})} \to 0$.

$\therefore \exists n_0 \in \mathbb{N}$:

$$n \geqq n_0 \Rightarrow d(x_n, x_{n+1}) \leqq \frac{1}{n^2}.$$

Let $m > n \geqq n_0$.

Then, $d(x_n, x_m)$

$$\leqq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m)$$

$$\leqq \sum_{l=n}^{\infty} \frac{1}{l^2} < \infty.$$

As $m, n \to \infty$, $d(x_n, x_m) \to 0$.

$\therefore \{x_n\} (\subset X)$ is a Cauchy seq. $\lrcorner$

Alternative proofs

1. Prove the following:

Let $T : X \to X$ be a contraction mapping with a parameter $r \in (0,1)$, where X is a metric space. Select $x \in X$ and $\varepsilon > 0$ that satisfy $d(x, Tx) < (1-r)\varepsilon$. Then, $B_\varepsilon(x)$ is T-invariant, where $B_\varepsilon(x) = \{z \in X : d(x,z) < \varepsilon\}$.

2. Let X be a metric space and $T : X \to X$ be a mapping that satisfies

$$d(Tx, Ty) \leq \frac{d(x,y)}{1 + d(x,y)}$$

for all $x, y \in X$. Let $x \in X$ and let $\varepsilon > 0$. Define $B_\varepsilon(x) = \{z \in X : d(x,z) < \varepsilon\}$. Under what condition, $B_\varepsilon(x)$ be T-invariant?

3. Prove the Banach contraction principle following the outline of the proof for φ-contractions.

4. Prove the fixed point theorem for Kannan mappings following the outline of the proof for φ-contractions.

5. Prove the Banach contraction principle following the outline of the proof for F-contractions.

6. Prove the fixed point theorem for Kannan mappings following the outline of the proof for F-contractions.