Cyclic contraction mappings

Def.

$X \neq \emptyset$

$A, B \subset X, \neq \emptyset$

$T : X \to X$ cyclic

$\Leftrightarrow T(A) \subset B,\ T(B) \subset A$

Def.

X MS

$A, B \subset X, \neq \emptyset$

$T : X \to X$ cyclic ρ-contraction

$\Leftrightarrow$ (1) T: cyclic

(2) $\exists \rho \in [0,1) : \forall x \in A, y \in B,$

$d(Tx, Ty) \leq \rho d(x, y)$

ex

$X = \mathbb{R}$

$A = (-\infty, 0]$

$B = [0, \infty)$

$T: X \to X$ defined by

$$Tx = -\frac{1}{2}x \quad (x \in \mathbb{R}).$$

Then, T is cyclic.

ex

$$Tx = \begin{cases} \frac{1}{2}x & \text{if } x \geqq 0; \\ -x & \text{if } x \leqq 0. \end{cases}$$

Let
$$\begin{cases} A = \left\{1, \frac{1}{2^2}, \frac{1}{2^4}, \cdots\right\} \\ B = \left\{\frac{1}{2}, \frac{1}{2^3}, \frac{1}{2^5}, \cdots\right\} \cup \{-1\} \\ X = A \cup B. \end{cases}$$

Then,

- T: cyclic
- $\forall x \in A,\ y \in B,\ d(Tx, Ty) \leqq \frac{1}{2} d(x, y)$.

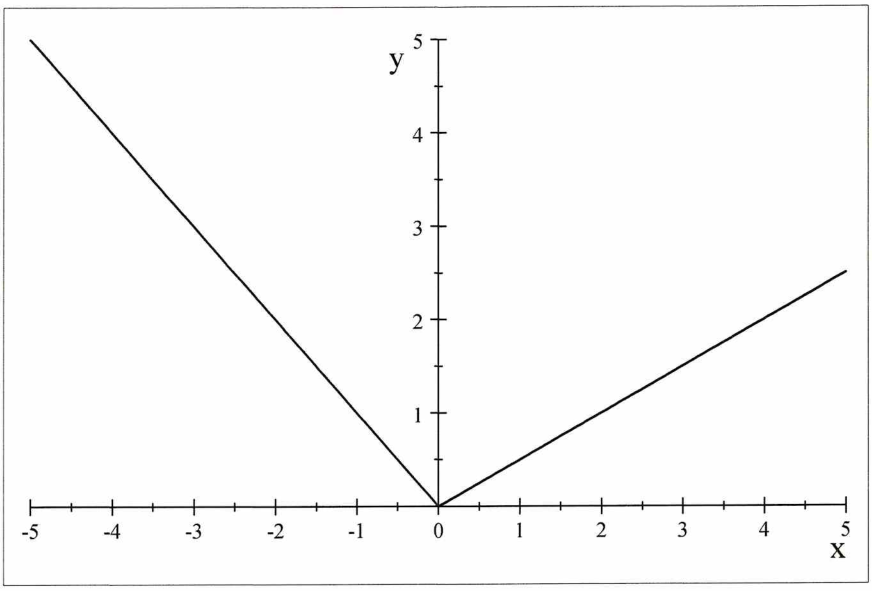
However,

- $\nexists \rho \in [0,1) : \forall x, y \in X,\ d(Tx, Ty) \leqq \rho\, d(x, y)$.

(∵) Set $x = -1 \in B$ and $y = \frac{1}{2^n} \in A \cup B$.

Then,
$$\frac{d(Tx, Ty)}{d(x, y)} = \frac{\left|\frac{1}{2^{n+1}} - 1\right|}{\left|\frac{1}{2^n} + 1\right|} \to 1$$
as $n \to \infty$.

$$Tx = \begin{cases} \frac{1}{2}x & \text{if } x \geq 0; \\ -x & \text{if } x \leq 0. \end{cases}$$



Lemma A

$$\boxed{\begin{array}{l} X \neq \emptyset \\ A, B \subset X, \neq \emptyset \\ X = A \cup B \\ T : X \to X \text{ cyclic} \\ \Rightarrow F(T) \subset A \cap B \end{array}}$$

Proof.

Let $x \in F(T)$ $(\subset X = A \cup B)$.

W.l.g., assume that $x \in A$.

As T is cyclic, $x = Tx \in B$.

Therefore, $x \in A \cap B$.

$\therefore F(T) \subset A \cap B$.

//

Lemma B.

$X \neq \emptyset$

$A, B \subset X, \neq \emptyset$

$A \cap B \neq \emptyset$

$T : X \to X$ cyclic

$\Rightarrow T(A \cap B) \subset A \cap B$

Proof.

Let $x \in A \cap B$.

We prove that $\underline{Tx \in A \cap B}$.

As $x \in A$ and T is cyclic, $Tx \in B$.

Similarly, as $x \in B$ and T is cyclic,

we have $Tx \in B$.

Hence, we obtain $Tx \in A \cap B$.

//

Lemma C.

X MS

$A, B \subset X \neq \phi$, closed

$T : X \to X$ cyclic

$x \in A \cup B$, $x_n = T^n x$ $(n \in \mathbb{N} \cup \{0\})$

$x_n \to u$

$\Rightarrow u \in A \cap B$

Proof:

W.l.g., assume that $x \in A$.

Then, $\begin{cases} \{x_{2i-1}\} \subset B \\ \{x_{2i}\} \subset A \quad \forall i \in \mathbb{N}. \end{cases}$

As $x_n \to u$, we have

$$\begin{cases} x_{2i-1} \to u \\ x_{2i} \to u. \end{cases}$$

As $\{x_{2i-1}\} \subset B$ and B is closed in X,

we have $u \in B$.

Similarly, $u \in A$.

Therefore, $u \in A \cap B$. //

Lemma

X MS
$T : X \to X$
$x \in X$, $x_n = T^n x$ $(\forall n \in \mathbb{N} \cup \{0\})$
$\exists \rho \in [0,1) : \forall n \in \mathbb{N}$,

$$d(x_n, x_{n+1}) \leq \rho\, d(x_{n-1}, x_n) \quad \text{—(*)}$$

$\Rightarrow \{x_n\}$ is a Cauchy sequence.

Proof.

From (*),
$$\begin{aligned} d(x_n, x_{n+1}) &\leq \rho\, d(x_{n-1}, x_n) \\ &\leq \rho^2 d(x_{n-2}, x_{n-1}) \\ &\cdots \\ &\leq \rho^n d(x_0, x_1). \quad \text{—(**)} \end{aligned}$$

Let $m, n \in \mathbb{N} : m \geq n$.
Using (**), we obtain

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \\ &\leq \rho^n d(x_0, x_1) + \cdots + \rho^{m-1} d(x_0, x_1) \\ &\leq \rho^n d(x_0, x_1)(1 + \rho + \rho^2 + \cdots) \\ &= \frac{1}{1-\rho} \rho^n d(x_0, x_1) \to 0. \end{aligned}$$

This shows that
(x_n) is a Cauchy sequence. //

Th

X CMS

$A, B \subset X$ $\neq \emptyset$, closed

$X = A \cup B$

$T : X \to X$ cyclic ρ-contraction

$\Rightarrow \exists 1\, x^* \in F(T) \subset A \cap B$

Proof.

Let $x \in X = A \cup B$ and

define $x_n = T^n x \in A \cup B$ $(\forall n \in \mathbb{N} \cup \{0\})$.

Observe that (x_n) is a Cauchy sequence.

To show that, we prove that

$$d(x_n, x_{n+1}) \leq \rho\, d(x_{n-1}, x_n).$$

Let $n \in \mathbb{N}$.

If $x_n \in A$, then $x_{n+1} = Tx_n \in B$.

If $x_n \in B$, then $x_{n+1} = Tx_n \in A$.

As T is a cyclical contraction,

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \rho\, d(x_{n-1}, x_n).$$

From Lemma, (x_n) is a Cauchy sequence. ⌟

As X is complete, $\exists u \in X : x_n \to u$.

From <u>Lemma C</u>, we have $u \in A \cap B$.

Therefore, $A \cap B \neq \emptyset$.

As $A, B (\subset X)$ are closed,

$A \cap B$ is also closed in X.

As X is complete, $A \cap B$ itself is
a complete metric space.

From <u>Lemma B</u>, $T : A \cap B \to A \cap B$ and

T is a ρ-contraction on $A \cap B$.

Thus, $\exists^1 x^* \in F(T)$ in $A \cap B$.

From <u>Lemma A</u>, T has no fixed point
outside of $A \cap B$.

Therefore, we have obtained
the desired result. //

Cor

X CMS

$T: X \to X$ contraction

$\Rightarrow \exists^1 x^* \in F(T)$

Proof.

Setting $A = B = X$ in Th,

we obtain the desired result.

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$A, B \neq \emptyset$

$S : A \to B$

$T : B \to A$

Define $G : A \cup B \to A \cup B$ as follows:

$$Gx = \begin{cases} Sx & \text{if } x \in A; \\ Tx & \text{if } x \in B. \end{cases}$$

$$\Rightarrow F(G) = F(S) \cap F(T) \subset A \cap B$$

<u>Proof</u>:

First, we show that <u>$F(G) \subset A \cap B$</u>.

Let $x \in F(G) \subset A \cup B$.

If $x \in A$, then $x = Gx = Sx \in B$.

Therefore, $x \in A \cap B$.

Similarly, if $x \in B$, then

$x = Gx = Tx \in A$.

$\therefore x \in A \cap B$.

We can conclude that $F(G) \subset A \cap B$. $\lrcorner$

Next, observe that

$F(G) = F(S) \cap F(T)$.

($\subset$) Let $x \in F(G) \subset A \cap B$.

As $x \in A$, we have $x = Gx = Sx$.

Similarly, as $x \in B$, $x = Gx = Tx$.

$\therefore x \in F(S) \cap F(T)$. ⌟

($\supset$) Let $x \in F(S) \cap F(T) \subset A \cap B$.

As $x \in A$, $x = Sx = Gx$.

Therefore, $x \in F(G)$.

$\therefore F(G) \supset F(S) \cap F(T)$. //

Cor

X CMS
$A, B \subset X \neq \phi$, closed
$X = A \cup B$
$S : A \to B$
$T : B \to A$
$\exists \rho \in [0,1) : \forall x \in A, y \in B,$
$$d(Sx, Ty) \leqq \rho\, d(x, y) \quad \text{— (*)}$$
$\Rightarrow \exists^1 x^* \in F(S) \cap F(T) \subset A \cap B$

Proof.
Define $G : X \to X$ as follows:
$$Gx = \begin{cases} Sx & \text{if } x \in A; \\ Tx & \text{if } x \in B. \end{cases}$$
Then, $F(G) = F(S) \cap F(T) \subset A \cap B$.
From (*), G is a cyclic ρ-contraction
From Th,
$$\exists^1 x^* \in F(G) = F(S) \cap F(T) \subset A \cap B.$$
//

Cyclic contraction mappings

1. Define a cyclic ρ-contraction mapping and present examples.

Prove the following assertions 1—7:

2. Let X be a nonempty set and let A, B be nonempty subsets of X. Suppose that $X = A \cup B$. Let $T : X \to X$ be a cyclic mapping. Then, $F(T) \subset A \cap B$.

3. Let X be a nonempty set and let A, B be nonempty subsets of X. Suppose that $A \cap B \neq \emptyset$. Let $T : X \to X$ be a cyclic mapping. Then, $T(A \cap B) \subset A \cap B$.

4. Let X be a metric space and let A, B be nonempty and closed subsets of X. Let $T : X \to X$ be a cyclic mapping. Let $x \in A \cup B$ and define $x_n = T^n x$ for all $n \in \mathbb{N}$. If $x_n \to u$, then $u \in A \cap B$.

5. Let X be a complete metric space and let A, B be nonempty and closed subsets of X. Suppose that $X = A \cup B$. Let $T : X \to X$ be a cyclic ρ-contraction mapping. Then, T has a unique fixed point in $A \cap B$.

6. Let A, B be nonempty sets, let $S : A \to B$, and let $T : B \to A$. Define $G : A \cup B \to A \cup B$ as follows:

$$Gx = \begin{cases} Sx & \text{if } x \in A; \\ Tx & \text{if } x \in B. \end{cases}$$

Then, $F(G) = F(S) \cap F(T) \subset A \cap B$.

7. <common fixed point theorem>

Let X be a complete metric space and let A, B be nonempty and closed subsets of X. Suppose that $X = A \cup B$. Let $S : A \to B$ and let $T : B \to A$. Suppose that there exists $\rho \in [0, 1)$ such that

$$d(Sx, Ty) \leq \rho d(x, y)$$

for all $x \in A$ and $y \in B$. Then, S and T has a unique common fixed point in $A \cap B$.

8. Devise your own theorem regarding a cyclic mapping and prove it.

References

[1] W.A. Kirk, P.S. Srinivasan, P. Veeramani, "Fixed point to mappings satisfyaing cyclical contractive conditions," Fixed Point Theory, 4.1 (2003), 79-89.

[2] P.S. Kumari and D. Panthi. "Cyclic contractions and fixed point theorems on various generating spaces," Fixed Point Theory and Applications 2015.1 (2015): 1-17.