Extensions of the Banach contraction principle

Th

X CMS

$T: X \to X$

$\exists \rho \in [0,1),\ \alpha, \beta, \gamma \in [0,1] : \alpha + \beta + \gamma = 1$

$\forall x, y \in X,$

$d(Tx, Ty)$

$\leq \rho[\alpha d(x,y) + \beta d(x, Tx) + \gamma d(y, Ty)]$

$\Rightarrow \exists 1\, x^* \in F(T) : \forall x \in X,\ T^n x \to x^*$

Proof

<Existence>

Let $x \in X$ and define $x_n = T^n x$ $(n \in \mathbb{N} \cup \{0\})$.

It follows that

$$
\begin{aligned}
d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
&\leq \rho[\alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, \underline{Tx_{n-1}}) \\
&\qquad + \gamma d(x_n, \underline{Tx_n})] \\
&= \rho[\alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, \underline{x_n}) + \gamma d(x_n, \underline{x_{n+1}})] \\
&= \rho(\alpha + \beta) d(x_{n-1}, x_n) + \rho\gamma d(x_n, x_{n+1}).
\end{aligned}
$$

Thus, we have

$$(1-\rho\gamma)d(x_n, x_{n+1}) \leqq \rho(\alpha+\beta)d(x_{n-1}, x_n).$$

$$\therefore d(x_n, x_{n+1}) \leqq \frac{\rho(\alpha+\beta)}{1-\rho\gamma} d(x_{n-1}, x_n). \quad -(*).$$

Define $\delta \equiv \dfrac{\rho(\alpha+\beta)}{1-\rho\gamma}$.

Then, $\delta \in [0, 1)$.

Indeed,

$$\begin{aligned} & \delta < 1 \\ \Longleftrightarrow\ & \rho(\alpha+\beta) < 1-\rho\gamma \\ \Longleftrightarrow\ & \rho(\alpha+\beta+\gamma) < 1 \\ \Longleftrightarrow\ & \rho < 1. \end{aligned}$$

From (*),

$$\begin{aligned} d(x_n, x_{n+1}) &\leqq \delta d(x_{n-1}, x_n) \\ &\leqq \delta^2 d(x_{n-2}, x_{n-1}) \\ &\leqq \cdots \\ &\leqq \delta^n d(x_0, x_1). \end{aligned}$$

We show that $\{x_n\}$ is a Cauchy sequence.

Let $m, n \in \mathbb{N} : m \geqq n$.

It follows that

$$
\begin{aligned}
& d(x_n, x_m) \\
& \leqq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \\
& \leqq \delta^n d(x_0, x_1) + \cdots + \delta^{m-1} d(x_0, x_1) \\
& \leqq \delta^n d(x_0, x_1)(1 + \delta + \delta^2 + \cdots) \\
& = \delta^n d(x_0, x_1) \frac{1}{1-\delta} \to 0 \quad \text{as } m, n \to \infty.
\end{aligned}
$$

As X is complete, $\exists x^* \in X : x_n \to x^*$.

$x^* = Tx^*$

It holds that

$$
\begin{aligned}
d(x^*, Tx^*) & \leqq d(x^*, x_{n+1}) + d(Tx_n, Tx^*) \\
& \leqq d(x^*, x_{n+1}) \\
& \quad + \rho[\alpha d(x_n, x^*) + \beta d(x_n, x_{n+1}) + \gamma d(x^*, Tx^*)]
\end{aligned}
$$

As $n \to \infty$, we obtain

$$d(x^*, Tx^*) \leqq \rho\gamma d(x^*, Tx^*).$$

$$\therefore (1 - \rho\gamma) d(x^*, Tx^*) \leqq 0.$$

$$\therefore x^* = Tx^*.$$

<Uniqueness>

Let $x^*, y^* \in F(T)$.

We have

$$d(x^*, y^*) = d(Tx^*, Ty^*)$$
$$\leq \rho\left[\alpha d(x^*, y^*) + \beta d(x^*, Tx^*) + \gamma d(y^*, Ty^*)\right]$$

$\therefore d(x^*, y^*) \leq \rho\alpha d(x^*, y^*)$.

$\therefore (1-\rho\alpha) d(x^*, y^*) \leq 0$.

$\therefore x^* = y^*$.

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$\beta = \gamma = 0$

Cor

X CMS

$T : X \to X$ a-contraction

i.e. $\exists a \in (0,1) : \forall x, y \in X,$

$d(Tx, Ty) \leq a\, d(x,y)$

$\Rightarrow \exists^1 x^* \in F(T) : \forall x \in X,\ T^n x \to x^*$

Cor

X CMS

$T: X \to X$ b-Kannan

i.e. $\exists b \in (0, \frac{1}{2}) : \forall x, y \in X,$

$$d(Tx, Ty) \leq b\big(d(x, Tx) + d(y, Ty)\big)$$

$\Rightarrow \exists^1 x^* \in F(T) : \forall x \in X,\ T^n x \to x^*$

Proof

It holds that

$$\begin{aligned} &d(Tx, Ty) \\ &\leq b\big(d(x, Tx) + d(y, Ty)\big) \\ &= 2b\left(\tfrac{1}{2} d(x, Tx) + \tfrac{1}{2} d(y, Ty)\right). \end{aligned}$$

Defining $\begin{cases} \rho = 2b \in (0,1) \\ \alpha = 0 \\ \beta = \gamma = 1/2 \end{cases}$,

we have

$$\begin{aligned} &d(Tx, Ty) \\ &\leq \rho\left[\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty)\right], \end{aligned}$$

where $\rho \in (0,1)$, $\alpha, \beta, \gamma \in [0,1] : \alpha + \beta + \gamma = 1$.

Thus, the desired results are obtained. //

Th.

X CMS
$T : X \to X$
$\exists \rho \in [0,1) : {}^\forall x, y \in X,$
$d(Tx, Ty)$
$\leq \rho \cdot \max\{d(x,y), d(x,Tx), d(y,Ty),$
$\frac{1}{2}[d(x,Ty)+d(Tx,y)]\}$
$\Rightarrow \exists^1 x^* \in F(T) : {}^\forall x \in X, T^n x \to x^*$

Proof.

<Existence>

Let $x \in X$ and define $x_n = T^n x$ $(n \in \mathbb{N} \cup \{0\})$.

We prove that (x_n) is a Cauchy sequence.

If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then

$$x_{n+2} = Tx_{n+1} = Tx_n = x_{n+1}.$$

Therefore, $x_n = x_{n+1} = x_{n+2} = \cdots$.

W.l.g., assume that

$$x_n \neq x_{n+1} \quad {}^\forall n \in \mathbb{N} \cup \{0\}.$$

Consequently, $d(x_n, x_{n+1}) > 0$.

It follows that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$

$$\leq \rho \cdot \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2}[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]\}$$

$$\leq \rho \cdot \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\}$$

$$\leq \rho \cdot \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \quad -(*)$$

Observe that $\underline{d(x_n, x_{n+1}) < d(x_{n-1}, x_n)}$. $-(**)$

If not, from $(*)$,

$$d(x_n, x_{n+1}) \leq \rho\, d(x_n, x_{n+1}) < d(x_n, x_{n+1})$$

$\rho \in [0,1)$
$d(x_n, x_{n+1}) > 0$

This is a contradiction.

Thus, $(**)$ holds.

From $(*)$ and $(**)$, we have

$$d(x_n, x_{n+1}) \leq \rho\, d(x_{n-1}, x_n)$$
$$\leq \rho^2 d(x_{n-2}, x_{n-1})$$
$$\cdots$$
$$\leq \rho^n d(x_0, x_1).$$

Let $m, n \in \mathbb{N} : m \geq n$.

It follows that

$d(x_n, x_m)$

$\leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m)$

$\leq \rho^n d(x_0, x_1) + \cdots + \rho^{m-1} d(x_0, x_1)$

$\leq \rho^n d(x_0, x_1)(1 + \rho + \rho^2 + \cdots)$

$= \rho^n d(x_0, x_1) \dfrac{1}{1-\rho} \to 0 \quad m, n \to \infty$.

This indicates that

$\{x_n\}$ is a Cauchy sequence.

As X is complete, $\exists x^* \in X : x_n \to x^*$.

$\underline{x^* = Tx^*}$

It holds that

$d(x^*, Tx^*)$

$\leq d(x^*, x_{n+1}) + d(Tx_n, Tx^*)$

$\leq d(x^*, x_{n+1})$

$\quad + \rho \cdot \max\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*),$

$\qquad \frac{1}{2}[\underline{d(x_n, Tx^*)} + d(x_{n+1}, x^*)]\}$

$\leq d(x^*, x_{n+1})$

$\quad + \rho \cdot \max\{\underline{d(x_n, x^*)}, d(x_n, x_{n+1}), \underline{d(x^*, Tx^*)},$

$\qquad \frac{1}{2}[\underline{d(x_n, x^*) + d(x^*, Tx^*)} + d(x_{n+1}, x^*)]\}$

$\leq d(x^*, x_{n+1})$

$\quad + \rho [d(x_n, x^*) + d(x_n, x_{n+1})$

$\qquad + d(x^*, Tx^*) + d(x_{n+1}, x^*)].$

Hence, $d(x^*, Tx^*) \leq \rho\, d(x^*, Tx^*)$ as $n \to \infty$.

$\therefore (1-\rho)\, d(x^*, Tx^*) \leq 0.$

As $1-\rho > 0$, we obtain $x^* = Tx^*$. $\lrcorner$

<Uniqueness>

Let $x^*, y^* \in F(T)$.

We have

$d(x^*, y^*)$

$= d(Tx^*, Ty^*)$

$\leqq \rho \cdot \max\{d(x^*, y^*), d(x^*, Tx^*), d(y^*, Ty^*),$

$\frac{1}{2}[d(x^*, Ty^*) + d(Tx^*, y^*)]\}$

$= \rho d(x^*, y^*)$.

Therefore, $(1-\rho) d(x^*, y^*) \leqq 0$.

As $1-\rho > 0$, we obtain $x^* = y^*$.

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Cor

X CMS

$T: X \to X$ Chatterjea

i.e. $\exists c \in [0, \frac{1}{2}) : \forall x, y \in X$

$$d(Tx, Ty) \leqq c\,(d(x, Ty) + d(Tx, y))$$

$\Rightarrow \exists^1 x^* \in F(T) : \forall x \in X,\ T^n x \to x^*$

Proof

Letting $\rho = 2c \in [0,1)$, we have

$$
\begin{aligned}
&d(Tx, Ty) \\
&\leqq c\,(d(x, Ty) + d(Tx, y)) \\
&= 2c \cdot \tfrac{1}{2}[d(x, Ty) + d(Tx, y)] \\
&\leqq \rho \cdot \max\{d(x,y), d(x,Tx), d(y,Ty), \\
&\qquad\qquad \tfrac{1}{2}[d(x,Ty) + d(Tx,y)]\}.
\end{aligned}
$$

Thus, we obtain the desired result. //

Cor

X CMS

$T : X \to X$

$\exists \rho \in [0,1) : {}^{\forall} x, y \in X,$

$d(Tx, Ty)$

$\leq \rho \cdot \max\{d(x,y), d(x,Tx), d(y,Ty)\}$

$\Rightarrow \exists 1\, x^* \in F(T) : {}^{\forall} x \in X,\ T^n x \to x^*$

Cor

X CMS

$T: X \to X$

$\exists \rho \in [0,1),\ \alpha, \beta, \gamma \in [0,1] : \alpha + \beta + \gamma = 1,$

$\forall x, y \in X,$

$$d(Tx, Ty) \leq \rho \left[\alpha d(x,y) + \beta d(x, Tx) + \gamma d(y, Ty) \right]$$

$\Rightarrow \exists^1 x^* \in F(T) : \forall x \in X,\ T^n x \to x^*$

Cor

X CMS

$T: X \to X$

$\exists \rho \in [0,1);\ \alpha, \beta, \gamma \in [0,1] : \alpha + \beta + \gamma = 1,$

$\forall x, y \in X,$

$$d(Tx, Ty) \leq \rho \left[d(x,y) \right]^{\alpha} \left[d(x, Tx) \right]^{\beta} \left[d(y, Ty) \right]^{\gamma}$$

$\Rightarrow \exists^1 x^* \in F(T) : \forall x \in X,\ T^n x \to x^*$

Cor

X CMS

$T : X \to X$

$\exists a \in [0,1),\ b \in [0, \frac{1}{2}) : \forall x, y \in X,$

(i) $d(Tx, Ty) \leq a\, d(x, y)$ **or**

(ii) $d(Tx, Ty)$

$$\leq b\big(d(x, Tx) + d(y, Ty)\big)$$

$\Rightarrow \exists^1 x^* \in F(T) : \forall x \in X,\ T^n x \to x^*$

Extensions of the Banach contraction principle

1. Prove the following Theorem 1:

Theorem 1. Let X be a complete metric space and let $T : X \to X$ be a mapping that satisfies

<T : Ćirić-Reich-Rus type contraction mapping>

$\exists\rho \in [0,1),\ \alpha,\beta,\gamma \in [0,1]$ such that $\alpha+\beta+\gamma = 1$,

$d(Tx,Ty) \le \rho[\alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty)]\ (\forall x,y \in X)$.

Prove that T has a unique fixed point $x^* \in F(T)$, and that $T^n x \to x^*$ (as $n \to \infty$) for any initial point $x \in X$.

2. Prove that fixed point theorems for contraction mappings and Kannan mappings can be derived from Theorem 1.

3. Prove the following Theorem A:

Theorem A. Let X be a complete metric space and let $T : X \to X$ be a mapping that satisfies the following:

$$\exists\rho \in [0,1) : \forall x,y \in X,$$
$$d(Tx,Ty) \le \rho\max\left\{d(x,y),\ d(x,Tx),\ d(y,Ty),\ \frac{1}{2}[d(x,Ty)+d(Tx,y)]\right\}.$$

Prove that T has a unique fixed point $x^* \in F(T)$, and that $T^n x \to x^*$ (as $n \to \infty$) for any initial point $x \in X$.

4. Show that Chatterjea's fixed point theorem is derived from Theorem A.

5. Verify that Theorem 1 and Theorem 2 below are derived from Theorem A.

Theorem 2. Let X be a complete metric space and let $T : X \to X$ be a mapping that satisfies the following:

$$\exists\rho \in [0,1),\ \alpha,\beta,\gamma \in [0,1] \text{ such that } \alpha+\beta+\gamma = 1,$$
$$d(Tx,Ty) \le \rho[d(x,y)]^{\alpha}[d(x,Tx)]^{\beta}[d(y,Ty)]^{\gamma}\ (\forall x,y \in X)$$

Prove that T has a unique fixed point $x^* \in F(T)$, and that $T^n x \to x^*$ (as $n \to \infty$) for any initial point $x \in X$.

6. Prove Theorem 2 directly without using Theorem A.

References

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