Contractive Mappings

X MS

$\{x_n\} \subset X$, $x \in X$

$\Rightarrow$ Equivalent

① $x_n \to x$

② $\forall \{x_{n_i}\} \subset \{x_n\}$, $x_{n_i} \to x$

③ $\forall \{x_{n_i}\} \subset \{x_n\}$, $\exists \{x_{n_j}\} \subset \{x_{n_i}\} : x_{n_j} \to x$

Proof

①$\Leftrightarrow$② OK

①$\Rightarrow$③ OK

③$\Rightarrow$①

Suppose by way of contradiction that $x_n \not\to x$.

Then, $\exists \varepsilon > 0 ; \forall n \in \mathbb{N}, \exists n' \geqq n : d(x_{n'}, x) \geqq \varepsilon$.

Thus, for $n=1$, $\exists n_1 \geqq 1 : d(x_{n_1}, x) \geqq \varepsilon$.

For $n = n_1 + 1 \in \mathbb{N}$, $\exists n_2 \geqq n_1 + 1 : d(x_{n_2}, x) \geqq \varepsilon$.

For $n = n_2 + 1 \in \mathbb{N}$, $\exists n_3 \geqq n_2 + 1 : d(x_{n_3}, x) \geqq \varepsilon$.

$\therefore \exists \{x_{n_i}\} \subset \{x_n\} : \forall i \in \mathbb{N}, d(x_{n_i}, x) \geqq \varepsilon$.

This contradicts ③.

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X MS

$\{x_n\}, \{y_n\} \subset X$

$\Rightarrow$ Equivalent

① $d(x_n, y_n) \to \infty$

② $\forall \{x_{n_i}\} \subset \{x_n\}, \{y_{n_i}\} \subset \{y_n\},$

$$d(x_{n_i}, y_{n_i}) \not\to \infty$$

Def.

X MS

X compact

$\Leftrightarrow \forall \{x_n\} \subset X, \exists \{x_{n_i}\} \subset \{x_n\}, x \in X : x_{n_i} \to x$

ex.

X MS

- $\{a, b, c\} \subset X$ compact
- $\{x_n\} \subset X : x_n \to x \in X$

$\Rightarrow \{x_n \mid n \in \mathbb{N}\} \cup \{x\}$: compact

X MS
$C \subset X$ compact
$\Rightarrow C$: closed in X.

<u>Proof</u>

Let $\{x_n\} \subset C : x_n \to x \in X$.

We show that <u>$x \in C$.</u>

As $\{x_n\} \subset C$ and C is compact,

$\exists \{x_{n_i}\} \subset \{x_n\}, y \in C : x_{n_i} \to y$.

As $x_n \to x$, we have $x_{n_i} \to x$.

Therefore, $y = x$.

We obtain $x (= y) \in C$.

//

X MS
$C \subset X$ compact
$\Rightarrow C$: bdd
i.e. $\exists M \geqq 0 : \forall x, y \in C,\ d(x,y) \leqq M$

<u>Proof</u>

Suppose by way of contradiction that C is not bdd.

Then, $\forall M \geqq 0,\ \exists x, y \in C : d(x,y) > M$.

$\therefore \forall n \in \mathbb{N},\ \exists x_n, y_n \in C : d(x_n, y_n) > n$.

$\therefore \exists \{x_n\}, \{y_n\} \subset C : d(x_n, y_n) \to \infty$. — (*)

As $\{x_n\}, \{y_n\} \subset C$ and C is compact,

$\exists \{x_{n_i}\} \subset \{x_n\},\ \{y_{n_i}\} \subset \{y_n\},\ x, y \in C :$
$x_{n_i} \to x,\ y_{n_i} \to y$.

Then, $d(x_{n_i}, y_{n_i}) \to d(x,y) \in \mathbb{R}$.

This contradicts (*).

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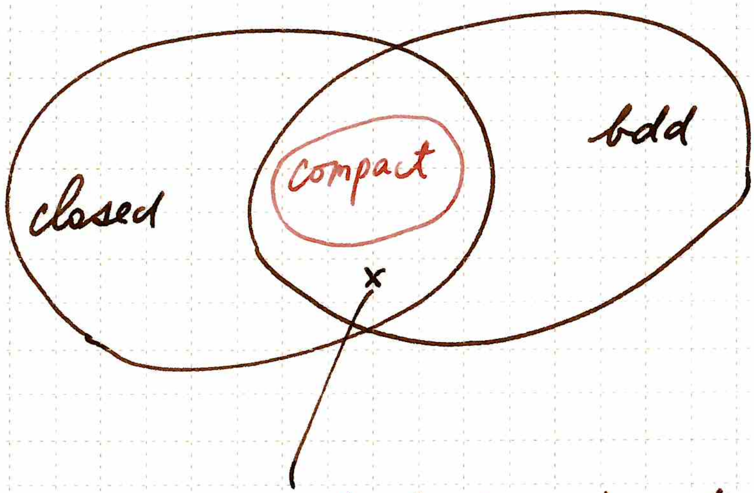
X : MS



$\mathbb{R}$ with the discrete metric

Def

$T : X \to Y$ <u>contractive</u>

$\Leftrightarrow \forall x, y \in X : x \neq y,$

$d(Tx, Ty) < d(x, y)$

ex

$Tx = \sin x$

$Tx = \cos x$

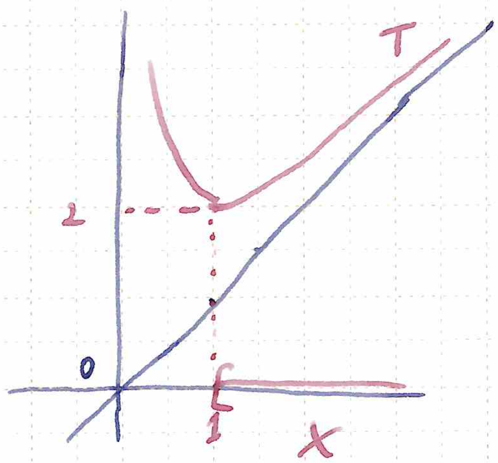
$Tx = \mathrm{Tan}^{-1} x$

where $X = \mathbb{R}$

ex

$X = [1, \infty)$ complete

$Tx = x + \frac{1}{x}$



Th

X compact metric space
$T : X \to X$ contractive
$\Rightarrow \exists^1 x^* \in F(T)$

Proof

<Existence>

Define $f : X \to \mathbb{R}$ by

$$f(x) = d(x, Tx) \quad \forall x \in X.$$

As f is conti. and X is compact,

$$\exists x^* \in X : f(x^*) = \inf_{z \in X} f(z). \qquad \text{—} \ (*)$$

$\underline{x^* = Tx^*}$

If $x^* \neq Tx^*$, then

$$f(Tx^*) = d(Tx^*, T^2 x^*)$$
$$< d(x^*, Tx^*) = f(x^*).$$

This contradicts $(*)$. ⌋

<Uniqueness>

Let $x^*, y^* \in F(T) : x^* \neq y^*$.

Then, $d(x^*, y^*)$

$$= d(Tx^*, Ty^*)$$
$$< d(x^*, y^*).$$

This is a contradiction.

//

Lemma

X metric space
$T : X \to X$ contractive
i.e. $u \neq v \Rightarrow d(Tu, Tv) < d(u, v)$ — (*)

$\exists x \in X$ s.t.
$x_n = T^n x \quad \forall n \in \mathbb{N} \cup \{0\}$
$x_{n_i} \to x^* \in X$ for some $\{x_{n_i}\} \subset \{x_n\}$
$\Rightarrow x^* \in F(T)$

Proof

If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then

$$x_{n+2} = Tx_{n+1} = Tx_n = x_{n+1}.$$

Thus, $x_n = x_{n+1} = x_{n+2} = \cdots$, and $x_n \in F(T)$.
In this case, the desired result follows.

Assume, w.l.g., that $x_n \neq x_{n+1} \ \forall n \in \mathbb{N} \cup \{0\}$.

$\{d(x_n, x_{n+1})\} (\subset \mathbb{R})$: convergent.

From (*),
$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\ &\overset{(*)}{<} d(x_n, x_{n+1}). \end{aligned}$$

This shows that $\{d(x_n, x_{n+1})\}$ is monotone decreasing, and hence, it is convergent. ⌟

$\underline{\lim d(x_n, x_{n+1}) = d(x^*, Tx^*)}$ —(**)

It follows that

$$
\begin{aligned}
&\lim d(x_n, x_{n+1}) \\
&\quad = \lim d(x_{n_i}, x_{n_i+1}) \\
&\quad = \lim d(x_{n_i}, Tx_{n_i}) \\
&\quad = d(x^*, Tx^*).
\end{aligned}
$$

$x_{n_i} \to x^*$
T: conti.

$\underline{d(x^*, Tx^*) = d(Tx^*, T^2x^*)}$ —(*3)

The following holds:

$$
\begin{aligned}
&d(x^*, Tx^*) \\
&\quad \overset{(**)}{=} \lim d(x_n, x_{n+1}) \\
&\quad = \lim d(x_{n+1}, x_{n+2}) \\
&\quad = \lim d(Tx_n, T^2x_n) \\
&\quad = \lim d(Tx_{n_i}, T^2x_{n_i}) \\
&\quad = d(Tx^*, T^2x^*).
\end{aligned}
$$

$x_{n_i} \to x^*$
T: conti.

ココまでは
T: NE でも言える。

$\underline{x^* = Tx^*}$

If $x^* \neq Tx^*$, then

$$
d(x^*, Tx^*) \underset{(*3)}{=} d(Tx^*, T^2x^*) \overset{(*)}{<} d(x^*, Tx^*)
$$

This is a contradiction.

//

Remark

X metric space
$T: X \to X$ conti.
$x \in X$
$x_n = T^n x \quad \forall n \in \mathbb{N} \cup \{0\}$
$x_n \to x^*$
$\Rightarrow x^* \in F(T)$

strong!

Proof

It holds that

$$
\begin{aligned}
Tx^* &= T(\lim x_n) \\
&= \lim Tx_n \\
&= \lim x_{n+1} \\
&= x^*.
\end{aligned}
$$

//

Remark

X MS

$T : X \to X$ NE

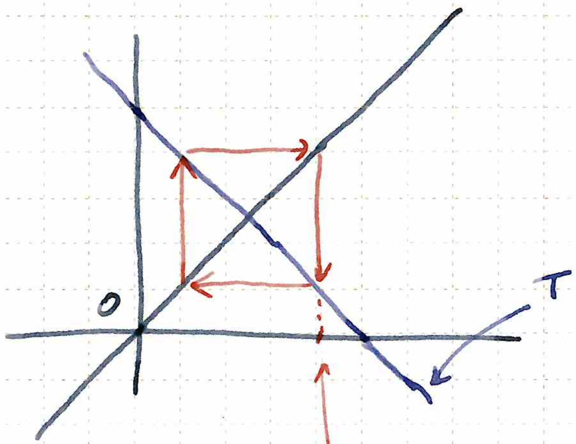
i.e. $d(Tu, Tv) \leq d(u,v) \quad \forall u, v \in X$ ← weak!

$x \in X$

$x_n = T^n x \quad \forall n \in \mathbb{N} \cup \{0\}$

$x_{n_i} \to x^* \in X$ for some $\{x_{n_i}\} \subset \{x_n\}$

$\not\Rightarrow x^* \in F(T)$



$x_{n_i} \to x^* \notin F(T)$

$\underline{\underline{\text{Th}}}$

$\boxed{\begin{array}{l} X \text{ compact metric space} \\ T: X \to X \text{ contractive} \\ \Rightarrow \exists 1\, x^* \in F(T) : \forall x \in X,\ T^n x \to x^* \end{array}}$

<u>Proof</u>

Let $x \in X$, and define $x_n = T^n x$ $\forall n \in \mathbb{N} \cup \{0\}$.

As X is compact,

$\exists \{x_{n_i}\} \subset \{x_n\},\ x^* \in X : x_{n_i} \to x^*$.

From <u>Lemma</u>, $x^* \in F(T)$.

As T is contractive, its fixed point is unique.

<u>$x_n \to x^*$</u>

i.e $\forall \{x_{n_j}\} \subset \{x_n\},\ \exists \{x_{n_k}\} \subset \{x_{n_j}\} : x_{n_k} \to x^*$.

As a fixed point of T is unique, from <u>Lemma</u>

we have the desired result.

//

Cor

$\mathbb{R}^N$

$C \subset \mathbb{R}^N$ $\neq \emptyset$, closed, bdd

$T: C \to C$ contractive

$\Rightarrow \exists^1 x^* \in F(T): {}^\forall x \in X,\ T^n x \to x^*$

Th

$C \subset \mathbb{R}^N \neq \emptyset$, closed

$T : C \to C$ contractive

$x^* \in F(T)$

$\Rightarrow$ (1) $F(T) = \{x^*\}$

(2) $\forall x \in C,\ T^n x \to x^*$

← C: bdd を落とせる.

Proof

As T is contractive, (1) is true.

We show (2).

Let $x \in X$, and define $x_n = T^n x \ \forall n \in \mathbb{N} \cup \{0\}$.

$\{x_n\} (\subset C)$: bdd

As $x^* = Tx^*$,

$$
\begin{aligned}
& d(x_n, x^*) \\
&= d(Tx_{n-1}, Tx^*) \\
&\leq d(x_{n-1}, x^*).
\end{aligned}
$$

T: NE

This shows that $\{x_n\}$ is bdd. ⌋

$\underline{x_n \to x^*}$

i.e. $\forall \{x_{n_i}\} \subset \{x_n\}, \exists \{x_{n_j}\} \subset \{x_{n_i}\} : x_{n_j} \to x^*$.

Let $\{x_{n_i}\} \subset \{x_n\}$.

As $\{x_{n_i}\}$ $(\subset \{x_n\} \subset C \subset \mathbb{R}^N)$ is bdd,

$\exists \{x_{n_j}\} \subset \{x_{n_i}\}, z \in \mathbb{R}^N : x_{n_j} \to z$.

As $\{x_{n_j}\} \subset C$, $x_{n_j} \to z$, and C $(\subset \mathbb{R}^N)$ is closed,

we obtain $z \in C$.

From <u>Lemma</u>, $z \in F(T) = \{x^*\}$. $\leftarrow$ T: contractive

$\therefore$ $x_{n_j} \to z = x^*$.

$\therefore$ $x_n \to x^*$.

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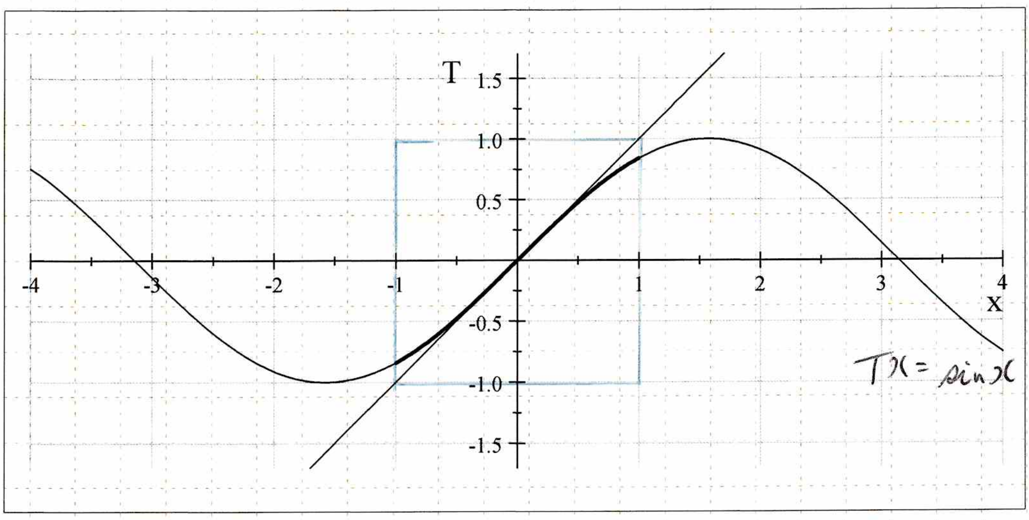
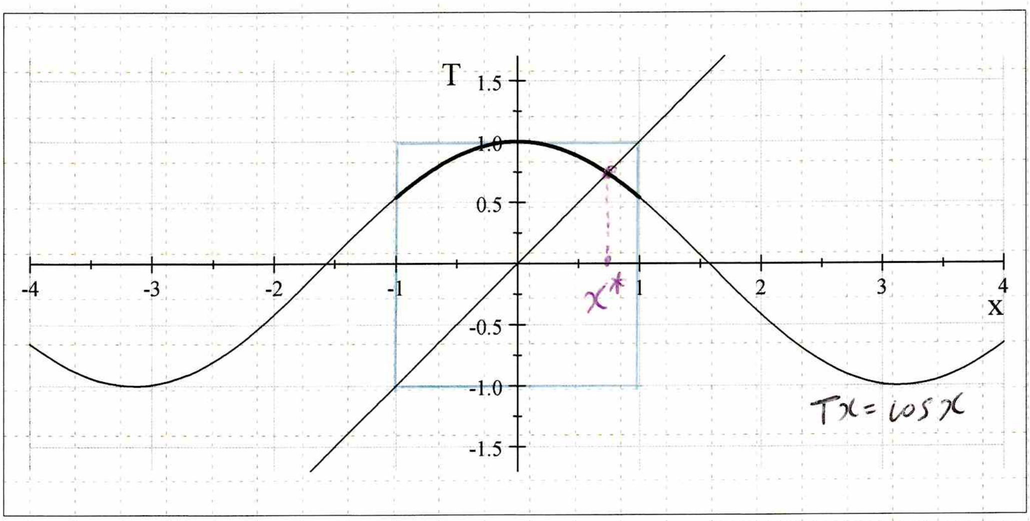
$Tx = \sin x$



$Tx = \cos x$



- $x_0 \in \mathbb{R}$

  $x_1 = \cos x_0 \quad \in [0,1]$

  $x_2 = \cos(\cos x_0)$

  $\vdots$

Then, $x_n \to x^* \in F(T)$.

Contractive mappings–Preliminaries

1. Show the following:

Let X be a metric space, let $\{x_n\}$ be sequence in X, and let $x \in X$. Then, the following three statements are equivalent:

(1) $x_n \to x$;

(2) For all subsequence $\{x_{n_i}\}$ of $\{x_n\}$, $x_{n_i} \to x$;

(3) For all subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \to x$.

2. Show the following:

Let X be a metric space and let $\{x_n\}$ and $\{y_n\}$ be sequences in X. Then, the following two statements are equivalent:

(1) $d(x_n, y_n) \to \infty$;

(2) For all subsequences $\{x_{n_i}\}$ of $\{x_n\}$ and $\{y_{n_i}\}$ of $\{y_n\}$, $d(x_{n_i}, y_{n_i}) \to \infty$.

3. Show the following:

Let X be a metric space and let C be a subset of X. If C is compact, then C is closed in X.

4. Show the following:

Let X be a metric space and let C be a subset of X. If C is compact, then C is bounded, that is,

$$\exists M \geq 0 \text{ such that } \forall x, y \in C,\ d(x,y) \leq M.$$

Contractive mappings

1. Show the following:

Let X and Y be metric spaces and let $T : X \to Y$ be a contractive mapping. Then, T is nonexpansive, that is,

$$d(Tx, Ty) \leq d(x,y).$$

2. Show the following:

Let X be a compact metric space, and let $T : X \to X$ be a contractive mapping. Then, there exists a unique fixed point x^* of T, and for any $x \in X$, the sequence $\{T^n x\}$ converges to x^*.

3. Show the following:

Let C be a closed subset of $\mathbb{R}^N$, let $T : C \to C$ be a contractive mapping, and let $x^* \in C$ be a fixed point of T. Then, (1) $F(T) = \{x^*\}$, and (2) for any $x \in X$, the sequence $\{T^n x\}$ converges to x^*.

Reference

[1] M. Edelstein, "On fixed and periodic points under contractive mappings," Journal of the London Mathematical Society 1.1 (1962): 74–79.