Mann Iteration for Contraction Mappings

X metric space
$C \subset X$ complete
$\Rightarrow C$: closed in X.

Proof

Let $\{x_n\} \subset C : x_n \to x \in X$.

We show that $\underline{x \in C}$.

As $\{x_n\}$ is convergent in X, it is a Cauchy seq. in X.

Therefore, $\{x_n\}$ is a Cauchy seq. in C.

As C is complete, $\exists y \in C : x_n \to y$.

As $x_n \to x \in X$ and $x_n \to y \in (C \subset) X$,

it holds that $x = y$.

Hence, $x\ (= y) \in C$.

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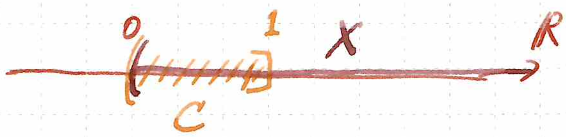
$\nLeftarrow$

$X = (0, \infty)\ (\subset \mathbb{R})$

$C = (0, 1]$



Then, C is closed in X.

However, C is <u>not</u> complete.

X CMS

$C \subset X$, $\neq \phi$

$\Rightarrow$ Equivalent

① C : complete

② C : closed in X.

Proof

① $\Rightarrow$ ② OK

② $\Rightarrow$ ①

Let $\{x_n\} (\subset C)$ be a Cauchy seq.

We show that $\exists x \in C : x_n \to x$.

As $\{x_n\} (\subset C) \subset X$ and X is complete,

$\exists x \in X : x_n \to x$.

As $\{x_n\} \subset C$ and $x_n \to x \in X$,

it follows from ② that $x \in C$.

$\therefore \exists x \in C : x_n \to x$.

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E <u>normed space</u>

$\Leftrightarrow$ (I) E : vector space

(II) $\|\cdot\| : E \to \mathbb{R}$

(N1) $\|x\| \geqq 0$; $\|x\| = 0 \Leftrightarrow x = 0$

(N2) $\|\alpha x\| = |\alpha| \|x\|$

(N3) $\|x+y\| \leqq \|x\| + \|y\|$

<u>Th</u>

$$\boxed{\begin{array}{l} E \text{ normed space} \\ d(x,y) = \|x-y\| \\ \Rightarrow (E, d) \text{ metric space} \end{array}}$$

E <u>Banach space</u>

$\Leftrightarrow$ (A) E : normed space

(B) (E, d) : complete

Th

$$\boxed{\begin{array}{l} E \text{ normed space} \\ d(x,y) := \|x-y\| \quad \forall x, y \in E \\ \Rightarrow (E, d) \text{ metric space} \end{array}}$$

Proof

(d1) $d(x,y) \geq 0$ OK. (N1)

$d(x,y) = 0 \Leftrightarrow x = y$

$$d(x,y) = \|x-y\| = 0 \Leftrightarrow x = y.$$

(d2) $d(x,y) = d(y,x)$

$$\begin{aligned} d(x,y) &= \|x-y\| \\ &= \|-(y-x)\| \\ &= |-1| \, \|y-x\| \\ &= \|y-x\| = d(y,x). \end{aligned}$$

(N2)

(d3) $d(x,y) \leq d(x,z) + d(z,y)$

$$\begin{aligned} d(x,y) &= \|x-y\| \\ &= \|x-z+z-y\| \\ &\leq \|x-z\| + \|z-y\| \\ &= d(x,z) + d(z,y) \end{aligned}$$

(N3)

$$\{\lambda_n\} \subset [0,1]$$

$$\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$$

$$A > 0$$

$$A(1-\lambda_n) < 1$$

$$\Rightarrow \prod_{i=1}^{\infty}\left(1-A(1-\lambda_i)\right) = 0$$

<u>Proof</u>

Define $P_n = \prod_{i=1}^{n}\left(1-A(1-\lambda_i)\right)$.

As $A(1-\lambda_i) < 1$, we have $P_n > 0$.

It follows that

$$\log P_n = \sum_{i=1}^{n} \log\left(1-A(1-\lambda_i)\right)$$

Remind that $\log(1-x) \leq -x \quad (\forall x < 1)$.

Letting $x = A(1-\lambda_i)\ (<1)$, we have

$$\log P_n = \sum_{i=1}^{n} \log\left(1-A(1-\lambda_i)\right)$$
$$\leq -A\sum_{i=1}^{n}(1-\lambda_i),$$

which yield that

$$0 < P_n \leq \exp\left(-A\sum_{i=1}^{n}(1-\lambda_i)\right)$$

As $A>0$ and $\sum_{i=1}^{\infty}(1-\lambda_i) = \infty$, we obtain $P_n \to 0$.

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Th

E Banach space
$C \subset E \neq \phi$, closed, convex
$T: C \to C$ r-contraction $(0<r<1)$
$\{\lambda_n\} \subset [0,1]$: $\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$

$$\begin{cases} x_1 \in C \\ x_{n+1} = \lambda_n x_n + (1-\lambda_n) T x_n \end{cases}$$

Mann iteration

$\Rightarrow x_n \to p \in F(T)$

Proof

As C is complete, $\exists^1 p \in F(T)$.
It holds that

$$\begin{aligned}
\|x_{n+1}-p\| &\leq \lambda_n \|x_n - p\| + (1-\lambda_n)\|Tx_n - p\| \\
&\leq \lambda_n \|x_n - p\| + (1-\lambda_n) r \|x_n - p\| \\
&= \{\lambda_n + (1-\lambda_n) r\} \|x_n - p\| \\
&= \{\lambda_n + r - \lambda_n r\} \|x_n - p\| \\
&= \{r - 1 + 1 + \lambda_n(1-r)\} \|x_n - p\| \\
&= \{1 - (1-r)(1-\lambda_n)\} \|x_n - p\| \\
&\leq \cdots \\
&\leq \prod_{i=1}^{n} \{1-(1-r)(1-\lambda_i)\} \|x_1 - p\|.
\end{aligned}$$

As $r \in (0,1)$ and $1-r>0$, $\prod_{i=1}^{\infty}(1-(1-r)(1-\lambda_i)) = 0$.

$\therefore x_n \to p$.

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Th

E Banach space
$C \subset E \neq \phi$, closed, convex
$T: C \to C$ r-contraction $(0<r<1)$
$\{\alpha_n\} \subset [0,1]$
$\{\lambda_n\} \subset [0,1]$: $\sum_{n=1}^{\infty}(1-\lambda_n)=\infty$

$$\begin{cases} x_1 \in C \\ y_n = \alpha_n x_n + (1-\alpha_n) T x_n \\ x_{n+1} = \lambda_n x_n + (1-\lambda_n) T y_n \end{cases}$$

Ishikawa iteration

$\Rightarrow x_n \to p \in F(T)$

Proof

As C is complete, $\exists^1 p \in F(T)$.
It holds that

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n x_n + (1-\alpha_n) T x_n - p\| \\ &= \|\alpha_n (x_n - p) + (1-\alpha_n)(T x_n - p)\| \\ &\leq \alpha_n \|x_n - p\| + (1-\alpha_n) \|T x_n - p\| \\ &\leq \alpha_n \|x_n - p\| + (1-\alpha_n) \|x_n - p\| r. \\ &= \{\alpha_n + r(1-\alpha_n)\} \|x_n - p\| \end{aligned}$$

Hence,

$$
\begin{aligned}
&\|x_{n+1}-p\| \\
&\quad = \|\lambda_n x_n + (1-\lambda_n) T y_n - p\| \\
&\quad \leq \lambda_n \|x_n - p\| + (1-\lambda_n)\|T y_n - p\| \\
&\quad \leq \lambda_n \|x_n - p\| + (1-\lambda_n)\, r\, \|y_n - p\| \\
&\quad \leq \lambda_n \|x_n - p\| + (1-\lambda_n)\, r\, \{\alpha_n + r(1-\alpha_n)\}\, \|x_n - p\| \\
&\quad = \{\lambda_n + (1-\lambda_n) r \alpha_n + (1-\lambda_n)(1-\alpha_n) r^2\} \|x_n - p\| \\
&\quad = \{1-(1-\lambda_n) + (1-\lambda_n) r \alpha_n + (1-\lambda_n)(1-\alpha_n) r^2\} \|x_n - p\| \\
&\quad = \{1-(1-\lambda_n)[1 - r\alpha_n - (1-\alpha_n) r^2]\} \|x_n - p\| \\
&\quad = \{1-(1-\lambda_n)(1 - r\alpha_n - r^2 + \alpha_n r^2)\} \|x_n - p\| \\
&\quad = \{1-(1-\lambda_n)(1 - r^2 - r\alpha_n(1-r))\} \|x_n - p\| \\
&\quad \leq \{1-(1-\lambda_n)(1 - r^2 - r(1-r))\} \|x_n - p\| \qquad \alpha_n = 1 \\
&\quad = \{1-(1-r)(1-\lambda_n)\} \|x_n - p\| \\
&\quad \leq \cdots \\
&\quad \leq \prod_{i=1}^{n} \{1-(1-r)(1-\lambda_i)\} \|x_1 - p\| \\
&\quad \longrightarrow 0.
\end{aligned}
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$\therefore x_n \to p.$

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$$\Big\downarrow \lambda_n = 1$$

Cor

E Banach space

$C \subset E \neq \emptyset$, closed, convex

$T : C \to C$ r-contraction

$\{\lambda_n\} \subset [0,1] : \sum_{n=1}^{\infty} (1-\lambda_n) = \infty$

$$\begin{cases} x_1 \in C \\ x_{n+1} = \lambda_n x_n + (1-\lambda_n) T x_n \end{cases}$$

$\Rightarrow x_n \to p \in F(T)$

$$\Big\downarrow \lambda_n = 0$$

Cor

E Banach space

$C \subset E \neq \emptyset$, closed, convex

$T : C \to C$ r-contraction

$\Rightarrow \forall x \in C,\ T^n x \to p \in F(T)$

Def.

$T: X \to X$ <u>Z-mapping</u>

$\Leftrightarrow {}^{\exists}a \in (0,1),\ b, c \in \left(0, \frac{1}{2}\right) :\ {}^{\forall}x, y \in C,$

(i) $d(Tx, Ty) \leqq a\, d(x, y)$,

(ii) $d(Tx, Ty) \leqq b\left(d(x, Tx) + d(y, Ty)\right)$, or

(iii) $d(Tx, Ty) \leqq c\left(d(x, Ty) + d(Tx, y)\right)$

X metric space

$T: X \to X$ Z-mapping

$\Rightarrow \exists \delta \in (0,1): \forall x \in X,\ p \in F(T),$

$$d(Tx, p) \leq \delta d(x, p)$$

<u>Proof</u>

As T is a Z-mapping

$\exists a \in (0,1),\ b, c \in \left(0, \frac{1}{2}\right): \forall x, y \in X,$

(i) $d(Tx, Ty) \leq a d(x, y)$,

(ii) $d(Tx, Ty) \leq b\left(d(x, Tx) + d(y, Ty)\right)$, or

(iii) $d(Tx, Ty) \leq c\left(d(x, Ty) + d(Tx, y)\right)$.

Define $\delta \equiv \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\} \in (0,1)$.

Let $x \in C$, $p \in F(T)$.

(i) In this case, we have

$$d(Tx, p) = d(Tx, Tp) \leq a d(x, p) \leq \delta d(x, p).$$

(ii) It holds that

$$\begin{aligned} d(Tx, p) &\leq b\left(d(x, Tx) + d(p, Tp)\right) \\ &\leq b d(x, p) + b d(p, Tx) \end{aligned}$$

$$\therefore (1-b) d(Tx, p) \leq b d(x, p)$$

$$\therefore d(Tx, p) \leq \frac{b}{1-b} d(x, p) \leq \delta d(x, p).$$

(iii) In this case,

$$
\begin{aligned}
&d(Tx, p) \\
&\quad \leqq c\bigl(d(x, Tp) + d(Tx, p)\bigr) \\
&\quad = c\,d(x, p) + c\,d(Tx, p)
\end{aligned}
$$

$$\therefore d(Tx, p) \leqq \frac{c}{1-c} d(x, p) \leqq \delta\, d(x, p).$$

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E Banach space

$C \subset E \neq \phi$, closed, convex

$T: C \to C$ Z-mapping

$\{\lambda_n\} \subset [0,1]$: $\sum_{n=1}^{\infty} (1-\lambda_n) = \infty$

$$\begin{cases} x_1 \in C \\ x_{n+1} = \lambda_n x_n + (1-\lambda_n) T x_n \end{cases}$$

$\Rightarrow x_n \to p \in F(T)$

Proof

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \lambda_n \|x_n - p\| + (1-\lambda_n) \|Tx_n - p\| \\
&\leq \lambda_n \|x_n - p\| + (1-\lambda_n) \delta \|x_n - p\| \\
&= (\lambda_n + \delta - \lambda_n \delta) \|x_n - p\| \\
&= (1 - 1 + \lambda_n + \delta - \lambda_n \delta) \|x_n - p\| \\
&= (1 - (1-\lambda_n)(1-\delta)) \|x_n - p\| \\
&\leq \cdots \\
&\leq \prod_{i=1}^{n} (1 - (1-\delta)(1-\lambda_i)) \|x_1 - p\|
\end{aligned}$$

As $1-\delta > 0$ and $(1-\delta)(1-\lambda_n) < 1$, we have

$x_n \to p$.

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More general iterations

Show the following:

1. Let X be a metric space and C be a complete subset of X. Then, C is closed in X.

2. Let X be a complete metric space and C be a nonempty subset of X. Then, the following two statements are equivalent.
(1) C is complete.
(2) C is closed in X.

3. Let $(E, \|\cdot\|)$ be a normed space. Define $d : E \times E \to \mathbb{R}$ as follows:

$$d(x,y) = \|x - y\| \quad \text{for all } x, y \in E.$$

Then, (E, d) is a metric space.

4. Let $\{\lambda_n\}$ be a sequence in the interval $[0, 1]$ such that $\sum_{n=1}^{\infty}(1 - \lambda_n) = \infty$. Let $A > 0$ such that $A(1 - \lambda_n) < 1$. Then, $\prod_{i=1}^{\infty}(1 - A(1 - \lambda_i)) = 0$.

5. Let E be a real Banach space and C be a nonempty, closed, and convex subset of E. Let T be an a-contraction mapping with $0 < a < 1$. Let $\{\lambda_n\}$ be a sequence in the interval $[0, 1]$ such that $\sum_{n=1}^{\infty}(1 - \lambda_n) = \infty$. Given $x_1 \in C$ arbitrarily, define a sequence $\{x_n\}$ by the following iteration scheme:

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n)Tx_n$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges to the unique fixed point p of T.

6. Let E be a real Banach space and C be a nonempty, closed, and convex subset of E. Let T be an b-Kannan mapping with $0 < b < 1/2$. Let $\{\lambda_n\}$ be a sequence in the interval $[0, 1]$ such that $\sum_{n=1}^{\infty}(1 - \lambda_n) = \infty$. Given $x_1 \in C$ arbitrarily, define a sequence $\{x_n\}$ by the following iteration scheme:

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n)Tx_n$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges to the unique fixed point p of T.

7. Let E be a real Banach space and C be a nonempty, closed, and convex subset of E. Let T be an b-Kannan mapping with $0 < b < 1/2$. Let $\{\lambda_n\}$ be a sequence in the interval $[0, 1]$ such that $\sum_{n=1}^{\infty}(1 - \lambda_n) = \infty$. Given $x_1 \in C$ arbitrarily, define a sequence $\{x_n\}$ by the following iteration scheme:

$$y_n = \lambda_n x_n + (1 - \lambda_n)Tx_n,$$
$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n)Ty_n$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges to the unique fixed point p of T.

References
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